ENLARGED SEPARATION PORTFOLIOS

FOR SYNTHESIZING AND ARBITRAGING SECURITIES

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Abstract

This paper seeks to provide a framework for separation portfolios when they are used not only as synthetics of a matching security but also as building blocks of arbitrage portfolios, in a background provided by the CAPM world. Firstly, synthetics are defined by means of a vectorial framework that maps portfolios onto their risk-return profiles. Separation portfolios are extensively analyzed afterwards, establishing three propositions that lay the groundwork for using them as synthetics. Next, a distinction is brought about between plain separation portfolios (which are located on the Capital Market Line) and enlarged separation portfolios (which lie outside the CML). Furthermore, it is shown that simple separation portfolios become synthetics in few cases only, whereas enlarged portfolios allow for synthetics in much wider contexts. Later, arbitrage portfolios are designed by means of simple separation portfolios in the context of the Security Market Line, and also by resorting to the enlarged ones without requiring the SML as a benchmark. Finally, the discussion extends over bond portfolios in the Arbitrage Pricing Theory (APT) environment, to embody their risky features into the viewpoint set forth in the paper.

JEL: G11, G12

Key Words: Plain Separation Portfolios, Enlarged Separation Portfolios, Synthetics, Portfolios, Arbitrage Portfolios

INTRODUCTION

It is usually understood that synthetic securities are patterns of cash flows built up from combining or decomposing sets of securities in order to replicate the cash flow streams of underlying real securities (background on Marshall and Kapner,1993). Financial engineers, drawing heavily from derivatives markets, widely resort to synthesizing securities and even portfolios to design new financial assets, and provide economic agents with risk-management shields.

In this paper, another sort of synthetics are focused on, namely those that can be attained when taking into account the risk-return profile of some asset or portfolio we wish to synthesize, or against which we look for an arbitrage opportunity, within the framework provided by some equilibrium model. The world of the Capital Asset Pricing Model will perform as the chosen benchmark, and an expansion on the Arbitrage Theory Model will be rendered later. (A classical development in Elton-Gruber, 1995; on this line of research, Apreda (2001a, 2001b, 2000) provides further details).

It is for section 1 to outline a vectorial framework into which risk-return profiles will be embedded eventually, and for section 2 to shape synthetic securities in such contexts. In section 3, simple separation portfolios perform as suitable synthetics in some cases, whereas in other cases we have to take up “enlarged” separation portfolios, as it will be showed in section 4. Through the next two sections, arbitrage environments will be expanded on, either with simple separation portfolios, or enlarged ones. Last of all, as the peculiar risky features of bond portfolios might cast doubts about the suitability of the CAPM to address bond arbitrage and separation portfolios, the paper shows that most of the preceding outcomes remain valid for bond portfolios by redefining the vectorial systematic risk-return profile in the context of the APT world. Conclusions will follow.

1. RISK-RETURN PROFILES WITHIN A VECTORIAL FRAMEWORK

Let us assume a horizon [t ; T], an initial level of wealth **w(t)** and a set **G**

$$\mathbf{G} = \{ \mathbf{G}_1, \mathbf{G}_2, \dots, \mathbf{G}_N \} = \{ \mathbf{G}_k \; ; \; k: 1, 2, 3, \dots, N \}$$

of distinctive financial assets, all of them available in the market at date “t”.

Generally speaking, by an arbitrary portfolio **P** is meant a vector whose components signal the proportion of initial wealth allocated eventually to each asset in **G** along [t ; T]. In other words,

$$\begin{cases} \mathbf{P} = \langle x_1 ; x_2 ; x_3 ; \dots\dots ; x_N \rangle \\ \text{where} \quad x_k = w(k, t) \; / \; w(t) \\ \text{means the proportion of asset } \mathbf{G}_k, \; k: 1, 2, \dots, N \end{cases} \tag{1}$$

We can see that any portfolio becomes assimilated to vectors or points in $\mathbf{R}^{N}$. Furthermore, let us call $\Pi \subseteq \mathbf{R}^{N}$ to the set of all vectors **P** for which (1) holds true. That is to say:

$$\Pi = \{ \mathbf{P} \in \mathbf{R}^{N} : \mathbf{P} \text{ fulfils (1)} \}$$

It was Markowitz (1952, 1959) and also Tobin (1958), the first ones to set forth a model of portfolio management within a mean-variance framework, while Sharpe (1963, 1964) was going to produce later the Security Market Line (SML) as a pricing device for securities when markets are in equilibrium. We are going to take advantage of both approaches in the following definition, so as to define a vectorial risk-return profile for securities and portfolios that will come in handy throughout this paper.

Definition 1

By the ***vectorial systematic risk-return profile of the portfolio P***

$$P = < x_1 ; x_2 ; x_3 ; \ldots\ldots\ldots\ldots ; x_N >$$

it is understood a vectorial map

$$F : \Pi \subseteq R^N \rightarrow R^2$$

$$F(P) = < \beta(P) ; E[R(P)] >$$

such that

$$\begin{cases} \beta(P) = \sum x_k \times \beta(G_k) \\ E[R(P)] = \sum x_k \times E[R(G_k)] \end{cases}$$

This vectorial function hangs on to Markowitz's risk-return framework of analysis, whereas the embedding of betas and expected returns will keep the discussion within the boundaries of the market model and the SML. This definition will be made fit to the APT's environment to deal with bonds portfolios in particular [Ross (1976)].

As we see from definition 1, two current algorithms are included that work out both the values of betas and expected returns of any portfolio **P**. As usual, the market model delivers beta as a weighted average of the betas of the portfolio's components.

Remarks

- *It must be borne in mind that some or almost all of the proportions* $\boldsymbol{x_k}$ *in (1) might be zero, this meaning that the investor did not allocate resources in the underlying assets.*

- *A single security* $\boldsymbol{A_k}$ *can also be regarded as a portfolio. In fact,*

$$A_k = < 0 ; 0 ; \ldots.. ; 1 ; \ldots. ; 0 >$$

  where the 1 stands in the k-th position.

- *As it will be shown in section 7, an extension to the world of Arbitrage Pricing Model is straightforward.*

For any portfolio **P**, we can also get access to the vectorial map **F** from a matricial perspective, by making use of two row vectors in $\mathbf{R}^N$:

i) a vector that gathers the financial assets betas:

$$< \beta(G_1), \beta(G_2), \beta(G_3), \ldots\ldots, \beta(G_N) >$$

ii) and another one consisting of the financial assets expected returns:

$$< E[R(G_1)], E[R(G_2)], E[R(G_3)], \ldots\ldots, E[R(G_N)] >$$

Therefore,

$$F(P) = F(< x_1 ; x_2 ; x_3 ; \ldots\ldots ; x_N >) =$$

$$= \begin{pmatrix} \beta(G_1) & \beta(G_2) & \beta(G_3) & \ldots\ldots & \beta(G_N) \\ E[R(G_1)] & E[R(G_2)] & E[R(G_3)] & \ldots\ldots & E[R(G_N)] \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ \ldots \\ \ldots \\ x_N \end{pmatrix}$$

that translates into

$$F(P) = F(< x_1 ; x_2 ; x_3 ; \ldots\ldots ; x_N >) =$$

$$= \begin{pmatrix} \sum x_k \times \beta(G_k) \\ \sum x_k \times E[R(G_k)] \end{pmatrix} = < \beta(P) ; E[R(P)] >$$

In all cases, index " k " runs from 1 to N within summation symbols.

2. SYNTHETIC SECURITIES WITHIN RISK-RETURN PROFILES

Let **A** be any financial asset or portfolio such that

$$F(A) = < \beta(A) ; E[R(A)] > \quad ; \quad A \in \Pi \subseteq R^N$$

We are moving forward so as to make explicit what a synthetic means in the context of this paper.

Definition 2

*By a **synthetic portfolio P of the asset A,** it is meant a portfolio on* Π,

$$P = < x_1 ; x_2 ; x_3 ; \ldots\ldots ; x_N >$$

so that its vectorial risk-return profile fulfills the following boundary condition:

$$F(P) = < \beta(P) ; E[R(P)] > = \begin{cases} \beta(P) = \beta(A) \\ E[R(P)] = E[R(A)] \end{cases}$$

Although mathematically we can find synthetics for **A** in almost any circumstances (in fact, we have to solve a system of two equations with **N** unknowns), it goes without saying that one thing is to get a theoretical solution and quite another to being able to come across with a down-to-earth synthetic, because of transaction costs and market microstructure features (Apreda, 2000a, 2000b, 2000c).

To solve this problem, separation portfolios consisting of a risk-free asset and the market portfolio (which amounts to index portfolios in real-life applications) seem suitable to a much wider extent than it is the case when trying with other sort of portfolios.

Whereas being a useful tool, plain separation portfolios fail to furnish synthetics in many cases. We can overcome such a hindrance by defining the more encompassing notion of enlarged separation portfolios, as it will be expanded on in section 4.

In the world of the Capital Market Line (CML), efficient portfolios consist of a free-risk asset **F** and the market portfolio **M**, provided equilibrium and homogeneous expectations are granted. They are called simple (or plain) separation portfolios and they exhibit the following structure:

Simple Separation Portfolio

$$\begin{cases} S = < x_F , x_M > \\ \text{subject to} \quad x_F + x_M = 1 \end{cases}$$

It can be shown (Blake, 2000; Apreda, 2001) that any separation portfolio fulfills the following relationship:

$$\beta(S) = x_M \tag{2}$$

and this amounts to saying that we hold a position in the risky portfolio equal to its measure of systematic risk.

3. SEPARATION PORTFOLIOS AS SYNTHETICS

In this section, we follow through the kind of synthetics that a simple separation portfolio could furnish, leaving for the next section the case of enlarged separation portfolios. The development will make apparent that getting access to simple or enlarged separation portfolios depends on whether security **A** belongs or not to the Security Market Line.

Proposition 1: If A belongs to the SML, then there is always a separation portfolio S such that

$$\beta(S) = \beta(A)$$

qualifying as a synthetic of A.

Proof: When **A** lies on the SML its risk-return profile comes out of

$$\mathbf{E[\,R(A)\,] = R(F) + < E[\,R(M)\,] - R(F) > \times \beta(A)}$$

On the other hand, as separation portfolios

$$\mathbf{S = < x_F \,;\, x_M >}$$

lie on the Capital Market Line, it holds by (2) that

$$\mathbf{\beta(S) = x_M}$$

then, by choosing

$$\mathbf{\beta(S) = \beta(A)}$$

it follows that

$$\mathbf{S = < x_F \,;\, x_M > = < 1 - \beta(S) \,;\, \beta(S) > = < 1 - \beta(A) \,;\, \beta(A) >}$$

which, in fact, qualifies as a good synthetic for **A** since

$$\mathbf{E[\,R(S)\,] = \sum x_k \times E[\,R(G_k)\,] = x_F \times R(F) + x_M \times E[\,R(M)\,]}$$

$$\mathbf{E[\,R(S)\,] = (\,1 - \beta(A)\,) \times R(F) + \beta(A) \times E[\,R(M)\,]}$$

$$\mathbf{E[\,R(S)\,] = R(F) + < E[\,R(M)\,] - R(F) > \times \beta(A) = E[\,R(A)\,]\,.} \quad \textit{END}$$

If **A** did not belong to the SML, and **S** is a separation portfolio, would it be feasible to find out a separation portfolio acting as synthetic for **A**? We now turn to the following proposition to make clear that such environment might not be attainable.

Proposition 2: ***If A does not belong to the SML, and S is a separation portfolio with the same expected return as A, it follows that***

$$\boldsymbol{\beta(S) \neq \beta(A)}$$

Proof: If **A** does not belong to the SML, then its expected return differs from the one ruled by the SML:

$$\mathbf{E[\,R(A)\,] \neq E[\,R(A)\,]_{SML}}$$

Let us suppose that

$$\mathbf{E[\,R(A)\,] > E[\,R(A)\,]_{SML}}$$

There are two separation portfolios relevant here: firstly, the one lying on the SML

$$\mathbf{S = < x_F \,;\, x_M >}$$

whose expected return is $\mathbf{E[\,R(A)\,]_{SML}}$. Furthermore, it has the same beta as **A**.

And secondly, we have to take into account the separation portfolio **S**’ whose expected return is

$$\mathbf{E[\,R(A)\,]}$$

Although **S**’ also belongs to the SML, it does so at the cost of having a higher beta. To prove this last statement, we shift to the capital market line (CML), where it holds

$$\begin{cases} E[\,R(S')\,] \;=\; R(F) \;+\; \{\; < E[\,R(M)\,] \;-\; R(F) > /\; \sigma(M) \;\} \times \; \sigma(S') \\ E[\,R(S)\,] \;=\; R(F) \;+\; \{\; < E[\,R(M)\,] \;-\; R(F) > /\; \sigma(M) \;\} \times \; \sigma(S) \end{cases}$$

Subtracting the second equation from the first one, and rearranging, we get:

$$\{\, E[\,R(S')\,] \;-\; E[\,R(S)\,]\,\} \;/\; \{\,\sigma(S') \;-\; \sigma(S)\;\} \;=\; \{\, E[\,R(M)\,] \;-\; R(F)\,\} \;/\,\sigma(M)\;\}$$

And solving for σ**(S')**

$$\{\, E[\,R(S')\,] \;-\; E[\,R(S)\,]\,\} \times \sigma(M) \;/\; \{\, E[\,R(M)\,] \;-\; R(F)\,\} \;+\; \sigma(S) \;=\; \sigma(S')$$

Moreover, for every separation portfolio lying on the CML

$$\sigma(P) \;=\; \beta(P) \times \sigma(M)$$

that can be replaced in the former relationship to get, after leaving out σ**(M)** from both sides:

$$\beta(S') \;=\; \beta(S) \;+\; \{\, E[\,R(S')\,] \;-\; E[\,R(S)\,]\,\} \;/\; \{\, E[\,R(M)\,] \;-\; R(F)\,\}$$

There being the differential rate of return between **S**' and **S** greater than zero, it follows

$$E[\,R(A)\,] > E[\,R(A)\,]_{SML} \quad\Rightarrow\quad \beta(S') \;>\; \beta(S)$$

In the same way,

$$[\,R(A)\,] \;<\; E[\,R(A)\,]_{SML} \quad\Rightarrow\quad \beta(S') \;<\; \beta(S)\,.$$ *END*

In the context of Proposition 2, it is worth highlighting two outcomes:

a) Provided security **A** does not belong to the SML, there will not be any separation portfolio acting as a synthetic of **A.**

b) If we had required that **A** and the separation portfolio **S** both share the same beta, then the expected return of **S** would not have matched that of **A**, because **S** is to lie on the SML**.**

4. ENLARGED SEPARATION PORTFOLIOS AS SYNTHETICS

To be a separation portfolio

$$S \;=\; < x_F \;;\; x_M >$$

means not only that we can set up a portfolio with only the risk-free asset and the market portfolio, but also the fulfillment of the boundary condition:

$$x_F \;+\; x_M \;=\; 1$$

Whenever this boundary condition does not hold, we are entitled to set forth the following definition.

Definition 3

*By **an enlarged separation portfolio** $\boldsymbol{S_e}$ it is understood any portfolio consisting of a risk-free asset and the market portfolio,*

$$S_e = < x_F \, ; \, x_M >$$

such that

$$x_F + x_M \neq 1$$

Proposition 2 showed that we could not fit a separation portfolio **S** with a financial asset **A**, the former becoming a synthetic of the latter, because their risk-return profile could not be the same. It is for the following proposition to redress that shortcoming and set up a synthetic for **A**.

Proposition 3: If A does not belong to the SML, we can find an enlarged separation portfolio to perform as a synthetic of A.

Proof: Let us suppose that **A** is a mispriced asset with respect to the SML, with a higher expected return than the predicted one

$$E[\,R(A)\,] \; > \; E[\,R(A)\,]_{SML} \quad \Rightarrow \quad E[\,R(A)\,] \; = \; E[\,R(A)\,]_{SML} \; + \; \alpha \quad , \quad \alpha > 0 \tag{3}$$

and we are going to build up a portfolio $\mathbf{S_e}$

$$S_e = < x_F \, ; \, x_M > \tag{4}$$

consisting in a free-risk asset and the market portfolio that will prove to become not only an enlarged separation portfolio but **A**'s synthetic as well.

Firstly, we choose a positive $\mathbf{x'_F}$ that solves

$$\alpha \; = \; x'_F \times R(F)$$

Secondly, let **S** be certain separation portfolio with the same beta as **A**, lying on the SML and fulfilling (2). That is to say:

$$\begin{cases} S \; = \; < x''_F \, ; \, x''_M > \; = \; < 1 - x''_M \, ; \, x''_M > \\ E[\,R(S)\,] \; = \; E[\,R(A)\,]_{SML} \; = \; x''_F \times R(F) \; + \; x''_M \times E[\,R(M)\,] \\ \beta(S) \; = \; \beta(A) \; = \; x''_M \end{cases} \tag{5}$$

Now we can make explicit the $\mathbf{S_e}$ structure in (4), by taking up:

$$S_e \; = \; < x_F \, ; \, x_M > \; = \; < x''_F + x'_F \, , \, x''_M > \tag{6}$$

We need to prove firstly that $\mathbf{S_e}$ is a synthetic of **A** and, secondly, a separation portfolio.

a) $\mathbf{S_e}$ is a synthetic of **A**.

By definition 1:

$$E[\,R(S_e)\,] \; = \; x_F \times R(F) \; + \; x_M \times E[\,R(M)\,]$$

by (6)

$$\mathbf{E[\,R(S_e)\,] = (\,x''_F + x'_F\,) \times R(F) + x''_M \times E[\,R(M)\,]}$$

by (5)

$$\mathbf{E[\,R(S_e)\,] = (\,1 - x''_M + x'_F\,) \times R(F) + x''_M \times E[\,R(M)\,]}$$

rearranging:

$$\mathbf{E[\,R(S_e)\,] = x'_F \times R(F) + R(F) + x''_M \times <E[\,R(M)\,] - R(F)>}$$

and by (3) and (5),

$$\mathbf{E[\,R(S_e)\,] = E[\,R(A)\,]_{SML} + \alpha = E[\,R(A)\,]} \qquad (7)$$

What is more,

$$\mathbf{\beta(S_e) = x_F \times \beta(F) + x_M \times \beta(M)}$$

$$\mathbf{\beta(S_e) = x_F \times 0 + \beta(A) \times 1 = \beta(A)}$$

So then, $\mathbf{S_e}$ is a synthetic for **A**

b) $\mathbf{S_e}$ is an enlarged separation portfolio since (6) holds,

$$\mathbf{S_e = <x_F\,;\,x_{M)}> = <x''_F + x'_F, x''_M>}$$

Now, by (5) and the fact that $\mathbf{x'_F > 0}$, we get

$$\mathbf{x''_F + x'_F + x''_M = 1 + x'_F > 1}$$

By the same token, if asset A were overpriced, we would arrive at the same conclusion, but in this case α **< 0** . *END*

In the former proposition an enlarged separation portfolio was designed perform as a synthetic, and by (6) and (7) it can be seen that it does not belong to the SML. In fact, there is a stronger result: no enlarged separation portfolio belongs to the SML, and the proof can be found in Appendix 1.

5. ARBITRAGE WITH SIMPLE SEPARATION PORTFOLIOS

Taking up the SML as a benchmark, let us assume we find out **A** to be a mispriced asset. This means that whereas the asset **A** should have in equilibrium the expected rate of return the SML assesses for its beta, **A** exhibits a higher or lower return than the one predicted. Without loss of generality, let us suppose the expected return of asset **A** is higher than expected. That is to say:

$$\mathbf{E[\,R(A)\,] > E[\,R(A)\,]_{SML} \Rightarrow E[\,R(A)\,] = E[\,R(A)\,]_{SML} + \alpha \;,\; \alpha > 0}$$

Asset **A** is cheaper than it should be. Hence, an arbitrage portfolio might be built up, by selling an asset or portfolio **S** lying on the SML with the followign features:

$$F(S) = < \beta(S) ; E[R(S)] > = \begin{cases} \beta(S) = \beta(A) \\ E[R(S)] = E[R(A)]_{SML} \end{cases} \tag{8}$$

Remark

- *If the expected return of asset A were lower than expected the discussion is similar, but* $\alpha < 0$

The conventional procedure consist of mixing two assets **B** and **C** on the SML so as to have

$$S = < x_B ; x_C >$$

such that (8) is fulfilled. Even so, assets the sort of **B** and **C** are not often available or the costs of running this portfolio **S** overruns the profit of the whole operation. Among the main sources of those transaction costs we have taxes, trading, information, microstructure and related financing costs. (details in Apreda, 2000)

Nonetheless, there is a simpler way to deal with this portfolio **S**. Instead of looking for distinctive assets lying on the SML we set up **S** as a separation portfolio

$$\begin{cases} S = < x_F ; x_M > \\ \text{where} \\ x_M = \beta(A) \end{cases} \tag{9}$$

We sell **S** so as to buy with that money asset **A,** thus setting up a self-financing portfolio.

$$\Delta P = < x_A ; x_S > = < +1 ; -1 >$$

By means of (2) and (9), we get

$$\beta(S) = \beta(A)$$

Let as work out the risk-return of ΔP :

$$E[R(\Delta P)] = x_A \times E[R(A)] + x_S \times E[R(S)]$$

$$E[R(\Delta P)] = 1 \times E[R(A)] + (-1) \times E[R(S)]$$

$$E[R(\Delta P)] = E[R(A)] - E[R(S)] > 0$$

On the other hand,

$$\beta(\Delta P) = x_A \times \beta(A) + x_S \times \beta(S)$$

$$\beta(\Delta P) = 1 \times \beta(A) + (-1) \times \beta(S) = 1 \times \beta(A) + (-1) \times \beta(A) = 0$$

Briefly: ΔP exhibits a positive return, with null systematic risk and it is self-financed. Therefore, it is an arbitrage portfolio.

6. ARBITRAGE WITH ENLARGED SEPARATION PORTFOLIOS

Although devising arbitrage portfolios against the Security Market Line provides with a wide range of applications, a concern often arises whether or not the SML is a feasible benchmark. At this point, enlarged portfolios have a say because they can make arbitrage attainable even when the SML fails to do so.

Let us picture a financial asset **A** with the following risk-return profile:

$$\mathbf{F(A) = < \beta(A) ; E[\, R(A)\,] >}$$

and we are interested to put forth a portfolio $\mathbf{S_e}$ consisting of a risk free asset and market portfolio so as to arbitrage **A**. For the time being, apart from sharing the same beta with **A**, no further assumptions will be made about $\mathbf{S_e}$.

However, so as to have an arbitrage portfolio $\Delta\mathbf{P}$, the structure of $\mathbf{S_e}$ must grant that at the end of the day there would be for $\Delta\mathbf{P}$ a positive differential return, null systematic risk and self financing.

Let us proceed by stages:

a) As the structure of $\mathbf{S_e}$ is given by:

$$\begin{cases} \mathbf{S_e = < x_F \; ; \; x_M >} \\ \quad \text{where} \\ \mathbf{x_M = \beta(A)} \end{cases} \tag{10}$$

To have $\mathbf{S_e}$ fully qualified we need to work out the value of $\mathbf{x_F}$. To begin with, we know that the expected return of **S** must meet the following conditions:

firstly, by definition 1:

$$\mathbf{E[\, R(S_e)\,] = x_F \times R(F) + x_M \times E[\, R(M)\,]} \tag{11}$$

secondly, by need of arbitrage: either **A** is overpriced or underpriced.

b) Let us assume that **A** is underpriced. Then,

$$\mathbf{E[\, R(A)\,] = E[\, R(S_e)\,] + \alpha \quad , \quad \alpha > 0}$$

Taking advantage of (11),

$$\mathbf{E[\, R(A)\,] = x_F \times R(F) + x_M \times E[\, R(M)\,] + \alpha \quad , \quad \alpha > 0} \tag{12}$$

c) In addition, we make:

$$\mathbf{\alpha = x'_F \times R(F)}$$

where $\mathbf{x'_F} > 0$, which allows to rewrite (12) and get

$$\mathbf{E[\, R(A)\,] = (\, x_F + x'_F\,) \times R(F) + x_M \times E[\, R(M)\,]} \tag{13}$$

Now, if it held that

$$\mathbf{x_F + x'_F + x_M = 1}$$

then, as $\mathbf{x_M = \beta(A)}$, it would follow that

$$\mathbf{E[\,R(A)\,] = (\,1 - \beta(A)\,) \times R(F) + \beta(A) \times E[\,R(M)\,]}$$

and this would lead to

$$\mathbf{E[\,R(A)\,] = R(F) + \beta(A) \times < E[\,R(M)\,] - R(F) >}$$

contrary to the assumption that A does not belong to the SML.
In addition,

$$\mathbf{x_F + x'_F + x_M \neq 1}$$

implies that

$$\mathbf{x_F + x_M = 1 - x'_F \neq 1} \qquad (14)$$

Solving for $\mathbf{x_F}$:

$$\mathbf{x_F = < 1 / R(F) > \times < E[\,R(A)\,] - x_M \times E[\,R(M)\,] > - x'_F}$$

If $\mathbf{x_F}$ were negative, then we take a loan at a cost of **R(F)**. We did not need to assume that short selling is allowed and not even that the risk-free rate for loans would be the same as when we buy the risk-free asset.

d) Finally, we have to show that $\mathbf{S_e}$ is an enlarged separation portfolio and that an arbitrage portfolio is feasible.

That $\mathbf{S_e}$ is an enlarged separation portfolio follows from the procedure developed above and (14).

Let us set up the following self-financing portfolio:

$$\mathbf{\Delta P = < x(A)\,;\, x(S_e) > = < +1\,;\, -1 >}$$

By relationships (10) and (12), we see it is a self-financing portfolio, with positive differential rate of return and null systematic risk. Therefore, it is an arbitrage portfolio.

7. THE CASE FOR BOND PORTFOLIOS

At least from a conceptual starting point, nothing would prevent the foregoing propositions to hold for quite general portfolios consisting of stocks, bonds and financial combos. There is a widespread contention, however, about using the CAPM world to bond pricing and arbitrage, unless strong qualifications or deep changes could be carried out on the whole context of this model application.

As Blake (2000) cleverly remarked, undiversifiable risk is not the only type of risk to deal with bonds. The interest-rate risk arising from coupons can have a great sway on the risk-return profile of bond portfolios.

Several risk-return metrics have been introduced to improve the pricing of bond portfolios. The most simple and straightforward one stems from the CAPM world and predicates the systemic risk on the grounds of the relative duration of the security against the duration of a market portfolio consisting of bonds only. Therefore, the expected return of a bond **B** comes out of

$$\mathbf{E[\,R(B)\,] = R(F) + < E[\,R(M)\,] - R(F) > \times \beta(B)}$$

where, in this particular setting,

$$\boldsymbol{\beta}\mathbf{(B) = Duration\ (B)\ \ /\ \ Duration\ (P_{M})}$$

In adopting this version for the CAPM world, the whole development that evolved through former sections runs quite well indeed. But this procedure comes at a cost, because of two features:

a) It is a recurrent observation in financial markets that the duration approach overstates the effect of short-term rates.

b) The relative duration fails with the term structure of rates of interest since it does not take into account non-parallel yield curve shifts.

A more substantive tool for coping with these shortcomings would be the adoption of the Arbitrage Pricing Theory Model (APT). In 1995, Elton, Gruber and Blake published in the Journal of Finance a relevant paper in which they developed relative pricing APT models that were successful in explaining expected returns in the bond market. They measured economic factors as changes in forecasts and make use of the following variables (Elton, Gruber and Blake, 1995) that arise when dealing with a multifactor return generating process whose main factors are:

a) market returns (the excess return on the market, net of the riskless rate);

b) default risk (absolute gap between corporate bonds and government bonds);

c) term risk (absolute gap between long-term and short-term government bonds);

d) unexpected changes in inflation;

e) unexpected changes in a measure of economic performance;

f) a measure of aggregate bond returns;

g) a measure of the return on mortgage securities relative to the return of government bonds.

So, an APT format seems significant when assessing expected returns of bond portfolios. And sheds light on the way we can proceed to embed this construct into the main body of this paper, a step we are going to carry out in next section.

7.1. A VECTORIAL RISK-RETURN PROFILE WITH A MULTIFACTOR MODEL

Let us assume that, as from now, we are interested in bond portfolios **P**, and that we choose a multifactor model with **L** distinctive orthogonal factors, albeit the viewpoint that follows is valid for quite general portfolios. If the factors were not orthogonal, a straightforward procedure produces an equivalent set of orthogonal ones (background on Elton-Gruber, 1995, chapter 8).

If we turn back to definition 1, we can easily reframe it to provide room to the underlying multifactor model underlying the APT world.

Definition 1a

*By **vectorial systematic risk-return profile of the bond portfolio P, in the APT world,***

$$P = < x_1 ; x_2 ; x_3 ; \dots\dots\dots\dots ; x_N >$$

it is understood a vectorial map

$$F : \Pi \subseteq R^N \rightarrow R^{L+1}$$

$$F(P) = < \beta_1(P); \beta_2(P); \beta_3(P); \dots\dots ; \beta_L(P); E[R(P)] >$$

such that

$$\begin{cases} \beta_1(P) = \sum x_k \times \beta_{1k}(G_k) \\ \beta_2(P) = \sum x_k \times \beta_{2k}(G_k) \\ \dots\dots\dots\dots\dots\dots\dots \\ \beta_L(P) = \sum x_k \times \beta_{Lk}(G_k) \\ E[R(P)] = \sum x_k \times E[R(G_k)] \end{cases}$$

On the same grounds, a synthetic asset or portfolio meets the next definition in this wider world of the Arbitrage Pricing Theory.

Definition 2a

*By a **synthetic portfolio P of the asset A,** in the APT world, it is meant a portfolio on Π,*

$$P = < x_1 ; x_2 ; x_3 ; \dots\dots ; x_N >$$

so that its vectorial systematic risk-return profile fulfills the following boundary condition:

$$F(P) = < \beta_1(P); \beta_2(P); \beta_3(P); \dots\dots ; \beta_L(P); E[R(P)] > =$$

$$= \begin{cases} \beta_1(P) = \beta_1(A) \\ \beta_2(P) = \beta_2(A) \\ \beta_3(P) = \beta_3(A) \\ \dots\dots \\ \beta_L(P) = \beta_L(A) \\ E[R(P)] = E[R(A)] \end{cases}$$

Next, let us assume that this APT world requires the following orthogonal indices or factors

$$\mathbf{I_1 ;\ I_2 ;\ I_3 ;\ \dots\dots\dots\ ;\ I_L}$$

As we know, in the standard APT setting, to each index there can be found an associated portfolio $\mathbf{P_k}$ such that

(15)

$$\begin{cases} \beta_j(P_k) = 1 \quad \text{if } k = j \\ \beta_j(P_k) = 0 \quad \text{if } k \neq j \ ; \ j : 1, 2, \dots\dots L \end{cases}$$

Now we proceed to set simple separation portfolios in this new setting, comprised of the risk-free asset and the orthogonal portfolios that belongs to the market portfolio. That is to say,

(16)

Simple separation portfolio in the APT world

$$\begin{cases} S = < x_F ;\ x_{P(1)} ;\ x_{P(2)} ;\ x_{P(3)} ;\ \dots\dots ; x_{P(L)} > \\ x_F + x_{P(1)} + x_{P(2)} + x_{P(3)} + \dots\dots\dots + x_{P(L)} = 1 \end{cases}$$

Let us assume that $x_{P(k)}$ stands for the proportion of wealth we allocate to the underlying portfolio to the orthogonal factor " $\mathbf{I_k}$ ".

It goes without saying that this sort of separation portfolio does not convey the features exhibited by separation portfolios in the CML world (for instance, to become the constituent elements of the efficient frontier).

Our next step is to figure out the betas of this simple separation portfolio, by resorting to properties of the multifactor model of generating returns and bearing in mind that the risk-free asset has a null beta.

(17)

$$\begin{cases} \beta_1(S) = \sum x_{P(j)} \beta_{j1}(P_j) = x_{P(1)} \\ \beta_2(S) = \sum x_{P(j)} \beta_{j2}(P_j) = x_{P(2)} \\ \beta_3(S) = \sum x_{P(j)} \beta_{j3}(P_j) = x_{P(3)} \\ \dots\dots\dots \\ \beta_L(S) = \sum x_{P(j)} \beta_{jL}(P_j) = x_{P(L)} \end{cases}$$

We wonder whether this framework of analysis furnishes with similar outcomes to those we got in propositions 1, 2 and 3 developed through sections 3 and 4, and also the arbitrage features that were expanded on along sections 5 and 6.

In fact, propositions 1 and 3 can be straightforwardly derived in this new setting, while proposition 2 cannot because of its strong dependence with the CML world. As for the arbitrage sections, their outcomes follow with reasonable changes.

For the sake of the argument, let us illustrate how to derive Proposition 1 and 3 in this multifactor environment.

Proposition 1a:

If the bond A belongs to the APT world, then there is always a simple separation portfolio S such that

$$< \beta_1(S); \beta_2(S); \beta_3(S); \dots\dots ; \beta_L(S) > \; = \; < \beta_1(A); \beta_2(A); \beta_3(A); \dots\dots ; \beta_L(A) >$$

qualifying as a synthetic of A.

Proof:

As **A** lies on the APT its expected returns turns out to be

$$E[\,R(A)\,] = R(F) + \beta_1(A) \times RP(1) + \beta_2(A) \times RP(2) + \dots\dots + \beta_L(A) \times RP(L)$$

where

$$RP(j) = E[\,R(P_j) - R(F)\,] \qquad ; \; j: 1, 2, \dots , L \tag{18}$$

Taking a separation portfolio

$$S = < x_F;\; x_{P(1)};\; x_{P(2)};\; x_{P(3)};\; \dots\dots\dots ;\, x_{P(L)} >$$

taking advantage of (17) we choose the proportions thereby:

$$\left\{ \begin{array}{l} \beta_1(S) = x_{P(1)} = \beta_1(A) \\ \beta_2(S) = x_{P(2)} = \beta_2(A) \\ \beta_3(S) = x_{P(3)} = \beta_3(A) \\ \qquad\qquad \dots\dots \\ \beta_L(S) = x_{P(L)} = \beta_L(A) \end{array} \right. \tag{19}$$

Therefore, the separation portfolio becomes

$$S = < x_F;\; \beta_1(A);\; \beta_2(A);\; \beta_3(A); \dots\dots ;\; \beta_L(A) >$$

where

$$x_F = 1 - \beta_1(A) - \beta_2(A) - \beta_3(A) - \dots\dots - \beta_L(A) \tag{20}$$

We only need to show that **S** performs as a synthetic of **A**.

The expected return of the simple separation portfolio it comes out of

$$E[\,R(S)\,] =$$

$$= x_F \times R(F) + x_{P(1)} \times E[R(P_1)\,] + x_{P(2)} \times E[R(P_2)\,] + \ldots + x_{P(L)} \times E[R(P_L)\,]$$

by (20) and (18) it follows that

$$E[\,R(S)\,] = E[\,R(A)\,]$$

Besides, by (19)

$$< \beta_1(S); \beta_2(S); \beta_3(S); \ldots\ldots ; \beta_L(S) > \; = \; < \beta_1(A); \beta_2(A); \beta_3(A); \ldots\ldots ; \beta_L(A) >$$

Summing up: S is a synthetic of A. *END*

In order to prove a similar statement to Proposition 3, we have to fashion the concept of enlarged separation portfolio into the APT's background.

Definition 3a

*By an **enlarged separation portfolio**, in the APT world, it is understood any portfolio whose structure takes the following format*

$$S_e = < x_F;\; x_{P(1)};\; x_{P(2)};\; x_{P(3)};\; \ldots\ldots ;\; x_{P(L)} >$$

such that

$$x_F + x_{P(1)} + x_{P(2)} + x_{P(3)} + \ldots + x_{P(L)} \neq 1$$

Proposition 3a:

If A does not belong to the APT, we can find an enlarged separation portfolio to perform as a synthetic of A.

Proof

Let us suppose that A is a mispriced asset with respect to the APT,with a higher return than the expected one. That is to say

$$E[\,R(A)\,] > E[\,R(A)\,]_{APT} \;\Rightarrow\; E[\,R(A)\,] = E[\,R(A)\,]_{APT} + \alpha \;, \quad \alpha > 0$$

On the other hand, betas of asset **A** are

$$< \beta_1(A);\; \beta_2(A);\; \beta_3(A); \ldots\ldots ;\; \beta_L(A) >$$

Firstly, we choose a positive $\mathbf{x'_F}$ that solves

$$\alpha = x'_F \times R(F)$$

Secondly, let **S** be a simple separation portfolio with the same beta as **A**, lying on the APT, with the following structure

$$S = < x''_F;\; x_{P(1)};\; x_{P(2)};\; x_{P(3)};\; \ldots\ldots ;\; x_{P(L)} >$$

and fulfilling

$$\begin{cases} \beta_1 (S) = x_{P(1)} = \beta_1 (A) \\ \beta_2 (S) = x_{P(2)} = \beta_2 (A) \\ \beta_3 (S) = x_{P(3)} = \beta_3 (A) \\ \quad \ldots\ldots\ldots \\ \beta_L (S) = x_{P(L)} = \beta_L (A) \end{cases}$$

Therefore,

$$E[R(S)] = E[R(A)]_{APT}$$

Now we can define $\mathbf{S_e}$ by means of the structure

$$S_e = < x'_F + x''_F ;\ x_{P(1)} ;\ x_{P(2)} ;\ x_{P(3)} ;\ \ldots\ldots\ldots ;\ x_{P(L)} >$$

We need to prove firstly that $\mathbf{S_e}$ is a synthetic of **A** and, secondly, an enlarged separation portfolio.

a) $\mathbf{S_e}$ is a synthetic of **A**.

The expected return of this portfolio directly follows from

$$E[R(S_e)] = (x'_F + x''_F) \times R(F) +$$

$$+ x_{P(1)} \times E[R(P_1)] + x_{P(2)} \times E[R(P_2)] + \ldots\ldots + x_{P(L)} \times E[R(P_L)]$$

rearranging and bearing in mind that **S** lies on the APT with the same betas as **A**

$$E[R(S_e)] = x'_F \times R(F) +$$

$$+ [1 - \beta_1 (A) - \beta_2 (A) - \beta_3 (A) - \ldots - \beta_L (A)] \times R(F) +$$

$$+ \beta_1 (A) \times E[R(P_1)] + \beta_2 (A) \times E[R(P_2)] + \ldots\ldots + \beta_L (A) \times E[R(P_L)]$$

finally

$$E[R(S_e)] = x'_F \times R(F) + E[R(A)]_{APT} = E[R(A)]$$

That $\mathbf{S_e}$ has the same betas as A follows from construction.

b) $\mathbf{S_e}$ is an enlarged separation portfolio since

$$x'_F + x''_F + x_{P(1)} + x_{P(2)} + x_{P(3)} + \ldots\ldots\ldots + x_{P(L)} = x'_F + 1 > 1$$

By the same token, if asset A were overpriced, we would arrive at the same conclusion, but in this case $\alpha < 0$. *END*

CONCLUSIONS

We can summarize the main outcomes of the paper in this way:

A vectorial framework from portfolios onto their systematic risk-return profiles seems functional to deal with arbitrage portfolios and synthetics.

Plain separation portfolios, arising from the Capital Market Line environment, allow designing synthetics only in a few cases. On the other hand, enlarged separation portfolios widen the scope for devising synthetics .

Plain separation portfolios lead to feasible arbitrage portfolios against the Security Market Line in few cases, whereas enlarged separation portfolios allow building up feasible arbitrage portfolios, even without the SML.

Finally, bond portfolios keep within the viewpoint introduced in this approach to plain and enlarged portfolios, and the only thing we have to do so as to take advantage of all the propositions proved in the paper is to fashion the vectorial systematic risk-return analysis to the world of the APT.

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APPENDIX 1

Lemma : An enlarged separation portfolio cannot lie on the Security Market Line.

Proof: An enlarged separation portfolio S_e *comes defined by the format*

$$\begin{cases} S_e = < x_F ; x_M > \\ \text{subject to: } x_F + x_M \neq 1 \end{cases}$$

Let us suppose that

$$x_F + x_M = 1 + \alpha \ , \quad \alpha > 0$$

and making

$$\alpha = x''_F \times R(F)$$

it follows that

$$x_F = 1 - x_M + x''_F \times R(F)$$

On the other hand, the expected return of S_e *comes assessed by*

$$E[\,R(S_e)\,] = x_F \times R(F) + x_M \times E[\,R(M)\,]$$

$$E[\,R(S_e)\,] = (\,1 - x_M + x''_F \times R(F)\,) \times R(F) + x_M \times E[\,R(M)\,]$$

and recalling that

$$x_M = \beta(\,S_e)$$

we get

$$E[\,R(S_e)\,] = x''_F \times R(F) + \{\,R(F) + < E[\,R(M)\,] - R(F) > \times \beta(\,S_e)\,\}$$

but the first term in the right hand side is greater than zero, preventing the enlarged separation portfolio from lying on the SML. The same procedure will hold if $\alpha < 0$. *END*