

A Note on Indirect Tax Evasion

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Abstract

This paper checks out the sensibility of indirect tax evasion models to assumptions about: i) tax function, ii) audit probability function, iii) risk aversion and iv) how tax evasion decisions occur. We found out that under perfect competition: i) the optimum level of tax evasion for ad valorem and specific taxes are equal but static comparative are not, ii) ad valorem tax evasion is lowest if the audit probability is decreasing with respect to the firm output, iii) risk aversion models produce similar results than direct cost of evasion models, but results are more conclusive when the risk is additive. JEL Classification: H25, H26.

Resumen

Este trabajo testea la sensibilidad de los resultados de los modelos de evasión para impuestos indirectos a los supuestos acerca de: i) el impuesto considerado, ii) la probabilidad de auditoría, iii) aversión al riesgo y iv) como sucede la evasión impositiva. Se demuestra que i) el nivel óptimo de evasión impositiva es similar bajo impuestos ad valorem o específicos, pero la estática comparativa no, ii) la probabilidad de auditoría decreciente respecto al nivel de producción es óptima, iii) los modelos de aversión al riesgo producen resultados similares a los que incluyen una función de costos de la evasión. JEL Classification: H25, H26.

1 Introduction¹

Although there is economic literature about tax evasion as old as one would look for, the issue has become a subfield of public finance only during the last 30 years. In their pioneering work, Allingham and Sandmo (1972) applied some, at that time, recent developments in the economics of uncertainty -Arrow (1970)- and the economics of crime -Becker (1968)- to analyze the individual taxpayer's decision on whether and to what extent to avoid the income tax by deliberate underreporting. That paper was concerned on direct taxes, those paying by an individual.

Researchers' interest on direct taxes, as opposed to indirect taxes, has been predominant and the literature is much wider.² This is so maybe reflecting that, empirically, direct taxes represent the bulk of developed countries tax revenues and, theoretically, in a microlevel analysis, any economic agent (such as a firm paying indirect taxes) could be reduced to an individual, the only decision makers that we could think off, and therefore, the direct tax evasion could be applied easily to an entrepreneur.

However, indirect tax evasion present many distinctive questions, but the research has been much fewer. Necessary references are Marelli (1984), Wang and Conant (1988), Virmani (1989), Yaniv (1988), Cremer and Ghavari (1994) and Yaniv (1995).

In our understanding, a key feature of this indirect tax evasion literature is that their results are very divergent according to each model's assumptions. In fact, some assumptions are not always explicit and it is not clear what would happen with results under different key assumptions. Many questions would arise and one could think off many different theoretical exercises.

In this paper we will do some of these exercises, changing key assumptions and checking out whether the results will change or not. In particular, we will focus on three issues that might be very relevant for public policy:

- A comparison among different indirect taxes on: i) Pareto optimum under tax evasion, ii) tax evasion equilibrium.

- Policy tools effect on the tax evasion optimum. Here we have at least three policy tools: tax rates, audit probabilities and fines. To study the effect of the former on tax evasion. is particularly interesting. On the other side, the effect of audit probability and fines are quite intuitive and they just have to be checked out to ensure models' consistency

- Separability between output and tax evasion decisions. This is a very important issue in order to evaluate the efficiency effects of tax evasion. From

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²Recent direct tax evasion literature reviews are Andreoni, Erard and Feistein (1998), Franzoni (1998), Alm (1999), Slemrod and Yitzhaki (2000) and Cowell (2003).

an economic point of view, the most pervasive effect of tax evasion is not the revenue lost by government (which is just a resources transfer from public sector to private) but the inefficiency implied when firms with higher production cost than average remain in the market because they conceal real revenues. However, this argument is valid only when there is any effect of tax evasion on output, which is not the case in most tax evasion models. As we will see, the exception is Virmani (1989), but its assumptions about the audit probability function make the model very unusual and it is difficult to generalize it.

In the first section, we will first summarize the literature. Next, we will present a basic model (the standard model) and then we will begin testing the results through changing assumptions in order to answer our questions. We close the paper with concluding remarks.

2 Background

Marrelli (1984) made the first application of the economics of uncertainty to indirect tax evasion. His objective was to study an entrepreneur's decision on whether, and to what extent, to avoid indirect taxes by underreporting, limiting the analysis to the case of a monopolistic firm and to ad valorem and profit taxes. In particular, the monopolistic market framework allowed him to study the degree of interdependence between the tax shifting and the tax evasion decisions, with risk averse entrepreneurs, both in the case of fixed and variable probability of being detected. The main, and surprising, Marrelli's result was that tax evasion has no influence on the shifting of the tax and, therefore, the after-tax marginal conditions for profit maximization are the same as those occurring in the absence of any evasion: the shifting and evasion problems are separable. The comparative static analysis was similar to the A-S model, but for the tax rate.

Marrelli examines also two alternative rules under which the probability of detection is a function of the tax base declared (increasing and decreasing functions) and he finds that the problems of tax shifting and tax evasion are no longer separable, the equilibrium quantity depends on the optimal interior rate of tax declaration, and vice versa. A decreasing function is more efficient, the monopolist both produces and declares more than in the case of a fixed probability of detection. The paper deals with sales tax (ad valorem tax), but there is also a comparison between a sales tax and a profit tax of equal yield. Marrelli shows that if we assume decreasing absolute risk aversion, an ad valorem tax is evaded as a percentage less than a profit tax of equal yield.

Same issues are examined under imperfect competition (duopoly with quantity setting firms) in Marrelli and Martina (1988). Three different tax functions are used (profit, ad valorem and specific tax) to study the impact of collusion and market shares on tax evasion. The model is built for risk averse taxpayers which have to decide on an amount to be concealed. In the latter, the paper

goes back to the original Allingham and Sandmo scheme. Results on separability are stronger than those of monopoly, since separability holds also on corner solutions. The origin of this is not duopoly but how the evasion is shaped (as an amount and not a tax due percentage). Comparative statics show similar qualitative results of policy parameters to those of monopoly.

Wang and Conant (1988) present a model for a risk averse monopolistic firm that can evade profit tax liability by cost overstatement. It is shown that neither the profit tax rate nor the penalty rate affect the profit maximizing rate of output. So, separability holds. The main difference with Marrelli (1984) is how tax evasion occurs, but assumptions and results are the same. Regarding to comparative statics analysis, in their model an increase in the tax rate and/or the probability of detection reduce the optimum level of tax evasion.

Yaniv (1988) is the first analysis of tax evasion within a withholding tax system, which account for the major part of income tax liability (and also of corporate taxes) in many countries. The analysis represents a mix between corporate and personal taxes. In this model, there are two kinds of tax evasion: i) the withholding tax system provides incentives for withholding agents to remit to the government less than the amounts withheld, ii) the conventional underreporting of the tax base (for non-withheld taxes). The model deals with a competitive and risk averse employer who is required to withhold a given proportion of total wage payments. There is also a profit tax, so it must be considered that understatement of wage payments results in overpayment of profit taxes. Main results are that optimal employment level is independent of fraud behaviour as long as the latter is optimal (separability) and an increase in the law enforcement parameters would discourage tax fraud. The analysis is very simple and some assumptions very strong: it assumes that withholding tax rates do not affect profits and that under both systems (with and without withholding regulations) audit probabilities are equal.

Virmani (1989) incorporates ad valorem tax evasion to the standard model of partial equilibrium in a competitive market, with free entry, U shaped average cost curves and risk neutral entrepreneurs. The model has two important characteristics: i) it assumes that probability of detection increases with firm output and ii) it includes a concealment cost function which depends on the proportion of sales declared. These two features made static comparative analysis quite different from previous results: evasion in this model is shown to be associated with production inefficiency (separability does not hold), evasion may increase with a rise in penalties and will be positively and production negatively related to tax rates (Laffer curve). Some papers after Virmani (1989) included the direct cost of evasion, and in fact, with risk neutral firms, it is necessary to get inner solutions. Separability does not hold because output level will affect the probability of detection and, therefore, tax evasion's pay off. Increasing probability of detection, although might be a common feature of real audit strategies, is a dominated strategy so they are not very common in the literature.³

³See Section 4.2.

Cremer and Ghavari (1994) present also a model of specific tax evasion but to study the more general issue of optimal taxation under tax evasion. The objective is to incorporate tax evasion in Ramsey's formulation of a tax structure. The idea behind is that any effective tax rate could be reached changing the tax code but also setting policy parameters (such as audit probability), so the Ramsey's problem of taxation can clearly be enriched considering tax evasion. Although the main objective are neither tax evasion optimum nor policy tool effects, these issues are treated. Market are competitive and, after Virmani (1989), it incorporates a direct cost of evasion function. Since this function is proportional to production and the audit probability is fixed, separability holds. Changes on the probability of detection and fines have the expected effect.

Yaniv (1995) presents a general model of tax evasion applicable to any type of tax that might be evaded by the firm, either through underreporting its tax base or overreporting allowed deductions. Firms are risk averse and face a proportional tax imposed on a certain tax base. As in Marrelli and Martina (1988) the activity decision is independent of the firm's attempt to evade taxes by misreporting, even for corner solutions, since the evasion decision is about an amount and not a proportion. An increase in the tax rate will always diminish the firm's statement deviation from the true value of its tax base, but we need to assume decreasing absolute risk aversion.

Summing up this literature, it is clear that key assumptions have very important effects on the results. In particular, assumptions made about:

- Market competition.
- Tax function.
- Concealment cost functions.
- Attitudes toward risk.
- Probability of detection function.
- Tax evasion as a percentage or an amount.

In the next Table, we present a summary of these differences.

	Tax	Audit Probability	Market	Risk	Evasion Direct Cost	Evasion
Marrelli (1984)	-Ad Valorem -Profit	-Fixed -Variable	-Monopoly	-Averse	No	Percentage
Wang and Conant (1988)*	-Profit	-Fixed	-Monopoly	-Averse	No	Percentage
Marelli and Martina (1988)	-Profit -Ad Valorem -Specific	-Fixed -Variable	-Duopoly	-Averse	No	Amount
Yaniv (1988)+	-Profit -Withholding	-Fixed	-Monopoly	-Averse	No	Percentage
Virmani (1989)	-Ad Valorem	-Variable	-Competitive	-Neutral	Yes	Percentage
Cremer y Ghavari (1994)	-Specific	-Fixed	-Competitive	-Neutral	Yes	Percentage
Yaniv (1995)	-Proportional on certain base	-Fixed	-Monopoly	-Averse	No	Amount

*Evasion trough cost overstatement.

+Evasion of tax withholdings.

3 Basic Model

We will present a model for the ad valorem tax, in a competitive market. The firm can underreport its sales in order to pay less tax. The government does not know real revenues, but it will audit several firms and tax evasion is punished with a fine. The firm is risk neutral. Tax evasion has a direct cost for the firm. Under these conditions, the firm will want to maximize expected profits:

$$Max_{\alpha, x} E[\Pi] = (1 - \phi)\Pi^{nd} + \phi\Pi^d \quad (1)$$

$$\Pi^{nd} = R - C - Rt\alpha - g(1 - \alpha)x(p) \quad (2)$$

$$\Pi^d = \Pi^{nd} - Rt(1 - \alpha)f \quad (3)$$

Where:

ϕ = Tax evasion detection probability (audit probability). We assume here that the audit rule is random (every firm faces the same audit probability).⁴

Π^{nd} = Profit if the tax evasion is not detected. It occurs with probability $(1 - \phi)$.

Π^d = Profit if tax evasion is detected. It occurs with probability ϕ .

$R = px(p)$ = Total Revenues.

⁴Random audit rules are obviously not very realistic. We are aware of the literature about audit rules, but we stick on this assumption in order to get clear cut results. See Arias (2004).

$C = cx(p) =$ Total Cost. We assume that firms exhibit constant marginal cost (c).

$c =$ Average and Marginal Cost.

$t =$ Sales tax (ad valorem).

$\alpha =$ Proportion of sales reported.

$g(1 - \alpha) =$ Direct tax evasion cost, which will depend on the proportion of sales reported. It is monotonic increasing and strictly convex function $g'(1 - \alpha) > 0$; $g''(1 - \alpha) > 0$. We could presume that $g(1) = \infty$ and $g(0) = 0$. To clarify, we could assume also that $g(1 - \alpha) = (1 - \alpha)G(1 - \alpha)$, where $G(1 - \alpha)$ is also a monotonic increasing and strictly convex function.

$f =$ Penalty ($f > 1$).

$x(p) =$ Output.

$p =$ Price.

Replacing (2) and (3) in the expected profit equation (1), we have:

$$\begin{aligned} E[\Pi] &= (1 - \phi)[R - C - Rt\alpha - g(1 - \alpha)x(p)] + \\ &+ \phi[R - C - Rt\alpha - g(1 - \alpha)x(p) - Rt(1 - \alpha)f] \end{aligned}$$

$$E[\Pi] = R - C - Rt\alpha - g(1 - \alpha)x(p) - \phi f[Rt(1 - \alpha)] \quad (4)$$

From this equation we can see that the firm will underreport its sales if and only if $\phi f < 1$; in any other case it is always profitable to report all the revenues (even if the concealment cost would be zero). Henceforth, we will assume:

$$\phi f < 1 \quad (5)$$

We could express equation (4) in unit terms:

$$E[\Pi] = [p - c - pt\alpha - g(1 - \alpha) - \phi[pt(1 - \alpha)f]]x(p)$$

This equation can be rewritten as:

$$E[\Pi] = [p - c - g(1 - \alpha) - pt[\alpha + \phi[(1 - \alpha)f]]]x(p) \quad (6)$$

We define,

$$w \equiv g(1 - \alpha) + pt[\alpha + \phi[(1 - \alpha)f]]$$

So, equation (6) is equal to:

$$E[\Pi] = [p - c - w]x(p)$$

Certainly, to maximize $E[\Pi]$ is similar to minimize w . As w is independent of $x(p)$, this implies that the production decision is independent of the tax evasion decision⁵, this is that stays the "separability" condition.

First order condition is:

$$\begin{aligned}\frac{\partial w}{\partial \alpha} &= -g'(1-\alpha) + pt - pt\phi f \leq 0 \\ g'(1-\alpha) &\leq pt(1-\phi f)\end{aligned}\quad (7)$$

So, the optimum α will be where the marginal cost of evasion is equal to the marginal benefit of tax evasion. In this case, the latter is fixed so, as long as $\phi f < 1$ there will be an inner solution.

Second order condition is:

$$\frac{\partial^2 w}{\partial \alpha^2} = g''(1-\alpha) > 0 \quad (8)$$

Which is consistent with the assumptions made regarding to the function $g(1-\alpha)$. Market equilibrium will be:

$$\begin{aligned}p(1-t^e) &= c + g(1-\alpha) \\ p &= \frac{c + g(1-\alpha)}{1-t^e}\end{aligned}\quad (9)$$

Where:

$$t^e = t[\alpha + \phi(1-\alpha)f] \quad (10)$$

In equation (9) g and t^e are evaluated at the optimal values of α . Total government revenues will be:

$$R = t^e p X(p)$$

3.1 Comparative Static Analysis

Differentiating equations (7), (9) and (10), and using the inequalities (5) and (8) we can obtain the following results of comparative statics with respect to the tax rate⁶:

$$\frac{\partial \alpha}{\partial t} = -\frac{p(1-\phi f)}{g''(1-\alpha)} < 0 \quad (11)$$

$$\frac{\partial t^e}{\partial t} = (\alpha + \phi(1-\alpha)f) - \frac{(1-\phi f)^2 pt}{g''(1-\alpha)} \geq 0 \quad (12)$$

$$\frac{\partial p}{\partial t} = \alpha + (1-\alpha)\phi f + (p-1)\frac{(1-\phi f)^2 pt}{g''(1-\alpha)} \geq 0 \quad (13)$$

⁵The inverse does not hold.

⁶See Appendix for detailed computations of these equations.

With respect to the audit probability:

$$\frac{\partial \alpha}{\partial \phi} = \frac{ptf}{g''(1-\alpha)} > 0 \quad (14)$$

$$\frac{\partial t^e}{\partial \phi} = \frac{(1-\phi f)pt^2 f}{g''(1-\alpha)} + tf(1-\alpha) > 0 \quad (15)$$

$$\frac{\partial p}{\partial \phi} = (1-p)\frac{(1-\phi f)pt^2 f}{g''(1-\alpha)} + tf(1-\alpha) \geq 0 \quad (16)$$

With respect to the fine:

$$\frac{\partial \alpha}{\partial f} = \frac{pt\phi}{g''(1-\alpha)} > 0 \quad (17)$$

$$\frac{\partial t^e}{\partial f} = \frac{(1-\phi f)pt^2 \phi}{g''(1-\alpha)} + (1-\alpha)\phi t > 0 \quad (18)$$

$$\frac{\partial p}{\partial f} = (1-p)\frac{(1-\phi f)pt^2 \phi}{g''(1-\alpha)} + (1-\alpha)\phi t \geq 0 \quad (19)$$

From this comparative analysis, we can draw the following conclusions:

- Public policy tools have the expected impact on tax evasion -equations (11), (14) and (17)-.

- The fine and the audit probability are perfect substitutes -equations (14)-(16) versus (17)-(19).

- Tax rates could have a negative impact on expected tax rates, since the positive effect on tax evasion (12).

- Tax rates could have also a negative impact on prices. Moreover, there is no any limit in the price elasticity of a tax rate change (equation 13).

- Audit probabilities and fines could have a positive or a negative impact on prices -equations (16) and (19).

We will expand on these results in the next section.

4 Extensions

4.1 Comparison among different tax functions

Under conditions of perfect competition and no tax evasion, specific and ad valorem taxes have identical effects, whereas under imperfect competition and

monopolies, specific taxes are generally dominated by ad valorem taxes, in terms of Pareto's optimum.⁷ Besides, with no tax evasion, there is a conventional view that profit taxes are neutral respect to a monopolist's profit maximizing rate of output, but ad valorem and specific tax are not.

It would be very interesting to check out whether these results remain under tax evasion. In order to do so, the results of optimal underreporting for different taxes of equal theoretical collection could be compared⁸.

At least, there could be applied three possible tax functions in order to compare production levels, tax evasion optimum and the separability condition under each tax function. Tax functions are:

Ad Valorem Tax: $t = tpx(p)$

Specific Tax: $t = tx(p)$

Profit Tax: $t = t(p - c)x(p)$

The ad valorem and the specific tax are available in any market, but the profit tax is meaningful only for monopoly and imperfect competition, since there are no profit under perfect competition. So possible comparisons are:

- i) Ad valorem vs specific tax under competition.
- ii) Ad valorem vs specific tax under monopoly.
- iii) Ad valorem vs profit tax under monopoly.
- iv) Specific tax vs profit tax under monopoly.

Some of these comparisons have been done by Marrelli (1984) and by Marrelli and Martina (1988). As we have seen, in both papers the comparison is not intended for perfect competition, but for monopoly and duopoly. Moreover, in those models taxpayer is risk averse and there is no a direct evasion cost function. So, in this document we will only do the comparison regarding to perfect competition.

4.1.1 Ad valorem versus specific tax under perfect competition

To begin with, we assume that fines, audit probability and the cost of the evasion are identical in both cases. Let us remember that, for the case of a specific tax, following Cremer and Ghavari, first order conditions imply:

$$g'(1 - \beta) = \tau(1 - \phi f) \tag{20}$$

Where:

$\tau =$ Specific tax.

⁷See i.e. Anderson, de Palma and Kreider (2000).

⁸Another criteria would be to compare different taxes of equal total tax revenues under the optimal tax evasion rates. Both are equally valid.

β = Proportion of sales reported.

And for ad valorem tax (equation 7):

$$g'(1 - \alpha) = pt(1 - \phi f)$$

In first place, we will fix the specific tax rate making it equal to $\bar{\tau}$; with which we have determined the theoretic collection in:

$$R = \bar{\tau}X(p)$$

Condition to meet is:

$$tpX(p) = \bar{\tau}X^*(p^*)$$

We know that $X^* = \sum x^*(p^*)$. Each $x^*(p^*)$ been quantities and prices of equilibrium. Since under perfect competition separability holds in both models, quantities and then prices are equal to X and p .

This implies that:

$$tp = \bar{\tau} \tag{21}$$

Observing the equations of determination of the optimal values of evasion under each tax, equations (7) and (20), this last equality implies that:

$$\alpha^* = \beta^* \tag{22}$$

Finally the price will also be the same one in both cases.

Comparing equations (11)-(16) with Cremer and Ghavari's equations (9a), (9b), (9c), (10a), (10b), (10c), we could see the qualitative differences in comparative statics between ad valorem and specific taxes:

	Specific Tax	Ad Valorem	Comparison
$\frac{\partial \alpha}{\partial t}$	< 0	< 0	=
$\frac{\partial t^e}{\partial t}$	≥ 0	≥ 0	=
$\frac{\partial p}{\partial t}$	> 0 ; < 1	≥ 0	\neq
$\frac{\partial \alpha}{\partial \phi}$	> 0	> 0	=
$\frac{\partial t^e}{\partial \phi}$	> 0	> 0	=
$\frac{\partial p}{\partial \phi}$	> 0	≥ 0	\neq

Discrepancies are the effect of the audit probability, fines and tax rates on prices. In fact, under the partial equilibrium model of ad valorem tax evasion, the most interesting result is that tax rates could have a negative impact on prices. The reason for this is that this effect depends on the effect of tax rates on i) evasion cost and ii) expected tax rate. The effect on the evasion cost is positive (the higher tax rate, the higher tax evasion and, therefore, the higher the per unit evasion cost). The effect on the expected tax rate is twofold: there is a direct and positive effect but there is also an indirect and negative effect, since a higher tax rate implies higher evasion and therefore a lower expected tax rate. So the final impact of the tax rate on expected tax rates will be uncertain.

Of course, this implies that the effect of tax rates on prices will also be uncertain, an increase on tax rates could imply a decrease on prices, as long as the increase on the evasion could overcome the increase on the tax evasion direct cost plus the direct effect of the tax rate on the expected tax rates. If we see equation (13), we will note that:

$$\begin{aligned} \text{if } p > 1 &\Rightarrow \frac{\partial p}{\partial t} > 0 \\ \text{if } p < 1 &\Rightarrow \frac{\partial p}{\partial t} \leq 0 \\ \text{if } p = 1 &\Rightarrow \frac{\partial p}{\partial t} = \alpha + (1 - \alpha)\phi f > 0 \end{aligned}$$

In the latter, the tax rate indirect effect on expected tax rate is equal to the effect on the evasion cost, so they compensate each other. This is the case for a commodity tax. The impact on the prices of the tax rate is always positive and less than 1 (Cremer and Ghavari's equation 9c: $\frac{\partial p}{\partial t} = \alpha + (1 - \alpha)\phi f$).

A very similar situation happens with the effect of audit probabilities and fines on prices. Since audit probability (and fines) have a direct and an indirect (through changes on tax evasion) effect on the expected tax rate, the final effect will be uncertain. In commodity tax, this effect is always positive (Cremer and Ghavari's equation 10c: $\frac{\partial p}{\partial \phi} = tf(1 - \alpha)$), since the indirect effect on the expected tax rate and the effect on evasion cost are equal.

4.2 Variable audit probability

The idea of a fixed audit probability is not very realistic. In fact, there is a specific literature subfield about audit rules in an strategic environment. Here, we will study the effects of a variable audit probability on the model results. We will assume $\phi(x)$. In this case, the expected profit function (??) is:

$$E[\Pi] = [p - c - pt\alpha - g(1 - \alpha) - \phi(x)[pt(1 - \alpha)f]]x(p) \quad (23)$$

Of course, in this model the condition of "separability" will not hold. Changes in the level of output will affect the expected tax rate and therefore the optimum level of tax evasion.

The firm will maximize equation (23) with respect to α and x . So, regarding to x :

$$\frac{\partial E[\Pi]}{\partial x} = p - c - g(1 - \alpha) - pt(\alpha + \phi(1 - \alpha)f - pt\phi'(1 - \alpha)fx(p)) \leq 0 \quad (24)$$

We define:

$$\begin{aligned} \phi_m &= \phi + \phi'x(p) \\ t_m^e &= t[\alpha + (1 - \alpha)\phi_m f] \end{aligned} \quad (25)$$

Replacing (25) in (24), first order condition will be:

$$\frac{\partial E[\Pi]}{\partial x} = p(1 - t_m^e) - c - g(1 - \alpha) \leq 0$$

And with respect to α :

$$\begin{aligned} \frac{\partial E[\Pi]}{\partial \alpha} &= [-pt + g'(1 - \alpha) + \phi f pt]x(p) \\ \frac{\partial E[\Pi]}{\partial \alpha} &= [g'(1 - \alpha) - pt(1 - \phi f)]x(p) \leq 0 \\ g'(1 - \alpha) &= pt(1 - \phi f) \end{aligned}$$

Market equilibrium will be:

$$p = \frac{c + g(1 - \alpha)}{1 - t_m^e} \quad (26)$$

An interesting question is to compare the level of tax evasion under this three configurations:

- i) α_1^* with $\phi' = 0$ (fixed audit probability)
- ii) α_2^* with $\phi' > 0$
- iii) α_3^* with $\phi' < 0$

For the first case, $t_m^e = t^e$ and equation (26) is equal to (10). In case (ii), $t_m^e > t^e$ and, since prices are exactly the same in both models, we need that $g(1 - \alpha_2^*) > g(1 - \alpha_1^*)$, which implies $\alpha_2^* < \alpha_1^*$: tax evasion will be greater under an increasing audit probability, than the tax evasion with fixed audit probability. Case (iii) is optimum from public policy, since tax evasion will be lower.

4.3 Attitudes toward risk

It is quite possible a straight forward application of Allingham and Sandmo model to tax evasion by firms. Risk aversion would generate an increasing cost of evasion (the risk premium) that does not exist for risk neutral agents, so there would not be a need for direct cost of evasion functions to get inner solutions. Results will be similar to A-S, and separability will hold under some specific conditions. This configuration allows to study the original A-S scheme where tax evasion was not proportional to the tax base, but an amount. We will do the exercise for both cases.

We need to assume that the Utility Function is concave ($U'(\Pi) > 0$ and $U''(\Pi) < 0$) and we will use the absolute Arrow-Pratt risk aversion measures:

$$R_A(\Pi^{nd}) = -\frac{U''(\Pi^{nd})}{U'(\Pi^{nd})}$$

$$R_A(\Pi^d) = -\frac{U''(\Pi^d)}{U'(\Pi^d)}$$

In particular, we need to assume that absolute risk aversion is decreasing with Π . So $R_A(\Pi^d) > R_A(\Pi^{nd})$.

4.3.1 Evasion as an amount

If available income is the only argument of the utility function, taxpayers maximize:

$$\begin{aligned} \text{Max}_\alpha E(U) &= (1 - \phi)U(\Pi^{nd}) + \phi U(\Pi^d) \\ \Pi^{nd} &= R - C - Rt + S \\ \Pi^d &= R - C - Rt + S - fS \end{aligned}$$

Where S is the total amount of taxes that the taxpayer conceals ($0 < S < Rt$). First and second order conditions are⁹:

$$\frac{(1 - \phi)}{\phi(f - 1)} \geq \frac{U'(\Pi^d)}{U'(\Pi^{nd})} \quad (27)$$

$$(1 - \phi)U''(\Pi^{nd}) + \phi(1 - f)^2U''(\Pi^d) \leq 0 \quad (28)$$

Second order condition is satisfied because of the assumptions we made about U . The comparative statics results are a little more difficult to obtain since it appears as an important element the shape of U . For example, if we would like to know the reaction of tax evasion from a change on tax rate, probabilities and fines we will find out:

⁹See computations in Appendix.

$$\frac{\partial S}{\partial t} = \frac{R}{D} \phi(1-f)U'(\Pi^{nd}) [R_A(\Pi^{nd}) - R_A(\Pi^d)] \leq 0 \quad (29)$$

$$\frac{\partial S}{\partial \phi} = -\frac{U'(\Pi^{nd}) - U(\Pi^d)(1-f)}{D} \leq 0 \quad (30)$$

$$\frac{\partial S}{\partial f} = \frac{U(\Pi^d)}{D} \leq 0 \quad (31)$$

Where D is the second order condition (equation 28). The first expression is negative if and only if we assume decreasing absolute risk aversion.¹⁰

4.3.2 Evasion as an percentage

Objective function is now:

$$\begin{aligned} Max_{\alpha} E(U) &= (1-\phi)U(\Pi^{nd}) + \phi U(\Pi^d) \\ \Pi^{nd} &= R - C - Rt\alpha \\ \Pi^d &= R - C - Rt\alpha - Rt(1-\alpha)f \end{aligned}$$

First and second order conditions are:

$$\frac{(1-\phi)}{\phi(f-1)} \geq \frac{U'(\Pi^d)}{U'(\Pi^{nd})} \quad (32)$$

$$(1-\phi)U''(\Pi^{nd})(-Rt)^2 + \phi U''(\Pi^d)(1-f)^2(-Rt)^2 \leq 0 \quad (33)$$

In this case, the effect on tax evasion of policy parameters are:

$$\frac{\partial \alpha}{\partial t} = -\frac{R^2 t}{D} \phi(1-f)U'(\Pi^{nd})R_A(\Pi^{nd})(\alpha + \dots + (1-\alpha)f) - R_A(\Pi^d)] \leq 0 \quad (34)$$

$$\frac{\partial \alpha}{\partial \phi} = -\frac{Rt}{D} [U'(\Pi^{nd}) - U'(\Pi^d)(1-f)] \geq 0 \quad (35)$$

$$\frac{\partial \alpha}{\partial f} = -\frac{U'(\Pi^d)Rt}{D} \geq 0 \quad (36)$$

An important results is that, when the risk is multiplicative and not additive, there is no a clear effect of the tax rate on the tax evasion behavior (equation 34) Remaining results are equal (equations 30 -31 versus 35 -36).

¹⁰Note that in this case $D \leq 0$; $(1-f) \leq 0$ and $[R_A(\Pi^{nd}) - R_A(\Pi^d)] < 0$.

5 Final Comments

Models presented here allow us to answer, partially, our questions:

- The firm's activity level is independent of the tax evasion (which is not the case for direct taxes and labor-supply decisions) only if we assume a fixed probability of detection.

- Risk aversion and concealment cost function are perfect substitutes in models.

- Ad valorem and unit taxes produces the same tax evasion optimum results under competitive markets. However, comparative statics results differs on the effects on prices.

- Decreasing probability of detection is optimum.

- Under risk aversion models, to get clear results about the reaction of tax evasion to a change on tax rates it is necessary to model the tax evasion as an amount.

The literature is growing fast (review by Cowell, 2004) and many models includes some issues that we have not touched, such as credit constrains, imperfect detection, corruption, agency problems within the tax administration, agency problems within the firm, strategic environment and audit rules, alternatives to the Expected Utility paradigm (like state dependent utility), arguments of the utility function (public goods), social norms and interdependence, temporal models, other kinds of uncertainty, etcetera.

There are other issues that remain almost untouched in the literature i.e. i) very weak analysis of withholding tax systems.¹¹ There is no analysis of withholding systems for indirect taxes. ii) The analysis of multiples and simultaneous corporate taxes is also very limited (profit, withholding and commodity), iii) welfare effects under tax evasion, iv) general equilibrium models.

Likewise, there are many of ways in which this paper could be extended. A comparison among different tax functions under other market configurations is probably the most interesting, but clearly not the only one.

¹¹Yaniv (1988) assumption that $\Pi_t = 0$ and the same probability of detection for employers and employees are very unrealistic.

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Appendix

Equation (11 , $\frac{\partial \alpha}{\partial t}$) Differentiating (7):

$$\begin{aligned} g'(1-\alpha) &= pt(1-\phi f) \\ \frac{\partial g'(1-\alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial t} &= p(1-\phi f) \\ \frac{\partial \alpha}{\partial t} &= -\frac{p(1-\phi f)}{g''(1-\alpha)} \end{aligned}$$

Equation (12 , $\frac{\partial t^e}{\partial t}$) From (10):

$$\begin{aligned} t^e &= t[\alpha + (1-\alpha)f\phi] = tA \\ \frac{\partial t^e}{\partial t} &= t'A + A't \\ A' &= \frac{\partial(\alpha + \phi(1-\alpha)f)}{\partial \alpha} \frac{\partial \alpha}{\partial t} \\ A' &= (1-\phi f)\left(-\frac{p(1-\phi f)}{g''(1-\alpha)}\right) \\ A' &= -p\frac{(1-\phi f)^2}{g''(1-\alpha)} \\ \frac{\partial t^e}{\partial t} &= \alpha + \phi(1-\alpha)f - \frac{(1-\phi f)^2 pt}{g''(1-\alpha)} \leq 0 \end{aligned}$$

Equation (13 , $\frac{\partial p}{\partial t}$) From (9):

$$\begin{aligned} \frac{\partial p}{\partial t} &= \frac{\partial g(1-\alpha)}{\partial t} + \frac{\partial t^e}{\partial t} \\ \frac{\partial g(1-\alpha)}{\partial t} &= \frac{\partial g(1-\alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial t} \\ &= g'(1-\alpha)\frac{p(1-\phi f)}{g''(1-\alpha)} > 0 \end{aligned}$$

$$\frac{\partial p}{\partial t} = g'(1-\alpha)\frac{p(1-\phi f)}{g''(1-\alpha)} + \alpha + \phi(1-\alpha)f - \frac{(1-\phi f)^2 pt}{g''(1-\alpha)}$$

Using equation (7):

$$\begin{aligned}\frac{\partial p}{\partial t} &= \frac{p^2(1-\phi f)^2 t}{g''(1-\alpha)} + \alpha + \phi(1-\alpha)f - \frac{(1-\phi f)^2 pt}{g''(1-\alpha)} \\ \frac{\partial p}{\partial t} &= (p-1)\frac{(1-\phi f)^2 pt}{g''(1-\alpha)} + (\alpha + \phi(1-\alpha)f)\end{aligned}$$

Equation (14 , $\frac{\partial \alpha}{\partial \phi}$). Differentiating (7)

$$\begin{aligned}\frac{\partial g'(1-\alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial \phi} &= -ptf \\ \frac{\partial \alpha}{\partial \phi} &= \frac{ptf}{g''(1-\alpha)}\end{aligned}$$

Equation (15 , $\frac{\partial t^e}{\partial \phi}$). Differentiating (10)

$$\begin{aligned}\frac{\partial t^e}{\partial \phi} &= t \left[\frac{\partial \alpha}{\partial \phi} + \frac{\partial(\phi f)}{\partial \phi} - \frac{\partial(\phi f \alpha)}{\partial \phi} \right] \\ \frac{\partial t^e}{\partial \phi} &= t \left[\frac{ptf}{g''(1-\alpha)} + f [1 - (\alpha' \phi + \alpha \phi')] \right] \\ \frac{\partial t^e}{\partial \phi} &= t \left[\frac{ptf}{g''(1-\alpha)} + f(1 - \frac{ptf\phi}{g''(1-\alpha)} - \alpha) \right] \\ \frac{\partial t^e}{\partial \phi} &= t \left[\frac{ptf}{g''(1-\alpha)} + f(1-\alpha) - \frac{ptf^2\phi}{g''(1-\alpha)} \right] \\ \frac{\partial t^e}{\partial \phi} &= \frac{(1-\phi f)pt^2 f}{g''(1-\alpha)} + tf(1-\alpha)\end{aligned}$$

Equation (16 , $\frac{\partial p}{\partial \phi}$). From (9)

$$\begin{aligned}\frac{\partial p}{\partial \phi} &= \frac{\partial g}{\partial \phi} + \frac{\partial t^e}{\partial \phi} \\ \frac{\partial p}{\partial \phi} &= -g'(1-\alpha)\frac{\partial \alpha}{\partial \phi} + \frac{\partial t^e}{\partial \phi} \\ \frac{\partial p}{\partial \phi} &= -g'(1-\alpha)\frac{ptf}{g''(1-\alpha)} + \frac{(1-\phi f)pt^2 f}{g''(1-\alpha)} + tf(1-\alpha) \\ \frac{\partial p}{\partial \phi} &= -pt(1-\phi f)\frac{ptf}{g''(1-\alpha)} + \frac{(1-\phi f)pt^2 f}{g''(1-\alpha)} + tf(1-\alpha) \\ \frac{\partial p}{\partial \phi} &= (1-p) \left[\frac{(1-\phi f)pt^2 f}{g''(1-\alpha)} \right] + tf(1-\alpha)\end{aligned}$$

Equation (17 , $\frac{\partial \alpha}{\partial f}$). From (7)

$$\begin{aligned}\frac{\partial g'(1-\alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial f} &= pt\phi \\ \frac{\partial \alpha}{\partial f} &= \frac{pt\phi}{g''(1-\alpha)}\end{aligned}$$

Equation (18 , $\frac{\partial t^e}{\partial f}$). From (10)

$$\begin{aligned}\frac{\partial t^e}{\partial f} &= t \left[\frac{\partial \alpha}{\partial f} + \frac{\partial(\phi f - \alpha \phi f)}{\partial f} \right] \\ \frac{\partial t^e}{\partial f} &= t \left[\frac{pt\phi}{g''(1-\alpha)} + \phi \frac{\partial(f - \alpha f)}{\partial f} \right] \\ \frac{\partial t^e}{\partial f} &= t \left[\frac{pt\phi}{g''(1-\alpha)} + \phi(1 - (\alpha' f + \alpha f')) \right] \\ \frac{\partial t^e}{\partial f} &= t \left[\frac{pt\phi}{g''(1-\alpha)} + \phi \left(1 - \left(\frac{pt\phi}{g''(1-\alpha)} f + \alpha \right) \right) \right] \\ \frac{\partial t^e}{\partial f} &= t \left[\frac{pt\phi}{g''(1-\alpha)} + (1-\alpha)\phi - \phi \left(\frac{pt\phi}{g''(1-\alpha)} f \right) \right] \\ \frac{\partial t^e}{\partial f} &= t \left[(1-\alpha)\phi + (1-\phi f) \frac{pt\phi}{g''(1-\alpha)} \right] \\ \frac{\partial t^e}{\partial f} &= (1-\alpha)\phi t + \frac{(1-\phi f)pt^2\phi}{g''(1-\alpha)}\end{aligned}$$

Equation (19 , $\frac{\partial p}{\partial f}$). From (10)

$$\begin{aligned}\frac{\partial p}{\partial f} &= \frac{\partial g}{\partial f} + \frac{\partial t^e}{\partial f} \\ \frac{\partial p}{\partial f} &= -g'(1-\alpha) \frac{\partial \alpha}{\partial f} + \frac{\partial t^e}{\partial f} \\ \frac{\partial p}{\partial f} &= -g'(1-\alpha) \frac{pt\phi}{g''(1-\alpha)} + (1-\alpha)\phi t + \frac{(1-\phi f)pt^2\phi}{g''(1-\alpha)} \\ \frac{\partial p}{\partial f} &= -pt(1-\phi f) \frac{pt\phi}{g''(1-\alpha)} + (1-\alpha)\phi t + \frac{(1-\phi f)pt^2\phi}{g''(1-\alpha)} \\ \frac{\partial p}{\partial f} &= (1-p) \frac{(1-\phi f)pt^2\phi}{g''(1-\alpha)} + (1-\alpha)\phi t\end{aligned}$$

Equation (27). First Order Condition

$$\begin{aligned}
\frac{\partial E(U)}{\partial S} &= (1 - \phi)U'(\Pi^{nd})\frac{\partial \Pi^{nd}}{\partial S} + \phi U'(\Pi^d)\frac{\partial \Pi^d}{\partial S} \leq 0 \\
\frac{\partial E(U)}{\partial S} &= (1 - \phi)U'(\Pi^{nd}) + \phi U'(\Pi^d)(1 - f) \leq 0 \quad (37) \\
-(1 - \phi)U'(\Pi^{nd}) &\geq \phi U'(\Pi^d)(1 - f) \\
\frac{(1 - \phi)}{\phi(f - 1)} &\geq \frac{U'(\Pi^d)}{U'(\Pi^{nd})}
\end{aligned}$$

Equation (28) . Second Order Condition:

$$\begin{aligned}
\frac{\partial^2 E(U)}{\partial S^2} &= (1 - \phi)U''(\Pi^{nd})\frac{\partial \Pi^{nd}}{\partial S} + \phi U''(\Pi^d)\frac{\partial \Pi^d}{\partial S}(1 - f) \leq 0 \\
\frac{\partial^2 E(U)}{\partial S^2} &= (1 - \phi)U''(\Pi^{nd}) + \phi(1 - f)^2 U''(\Pi^d) \leq 0
\end{aligned}$$

Equation (29 , $\frac{\partial S}{\partial t}$). Total differentiating FOC (37)

$$\begin{aligned}
d &= (1 - \phi)U'(\Pi^{nd}) + \phi U'(\Pi^d)(1 - f) \\
\frac{\partial S}{\partial t} &= -\frac{d_t}{d_S} \\
d_t &= (1 - \phi)U''(\Pi^{nd})(-R) + \phi U''(\Pi^d)(1 - f)(-R) \\
d_S &= D \\
\frac{\partial S}{\partial t} &= -\frac{(1 - \phi)U''(\Pi^{nd})(-R) + \phi U''(\Pi^d)(1 - f)(-R)}{D} \\
\frac{\partial S}{\partial t} &= \frac{R}{D} [\phi(1 - f)U''(\Pi^d) + (1 - \phi)U''(\Pi^{nd})]
\end{aligned}$$

Where D is the second order condition. We know from (37) that:

$$(1 - \phi) = -\phi(1 - f)\frac{U'(\Pi^d)}{U'(\Pi^{nd})}$$

Replacing:

$$\begin{aligned}
\frac{\partial S}{\partial t} &= \frac{R}{D} \left[\phi(1 - f)U''(\Pi^d) - \phi(1 - f)\frac{U'(\Pi^d)}{U'(\Pi^{nd})}U''(\Pi^{nd}) \right] \\
\frac{\partial S}{\partial t} &= \frac{R}{D}\phi(1 - f)U'(\Pi^d) \left[\frac{U''(\Pi^d)}{U'(\Pi^d)} - \frac{U'(\Pi^{nd})}{U'(\Pi^{nd})} \right] \\
\frac{\partial S}{\partial t} &= \frac{R}{D}\phi(1 - f)U'(\Pi^{nd}) [R_A(\Pi^{nd}) - R_A(\Pi^d)] \leq 0
\end{aligned}$$

Equation (30 , $\frac{\partial S}{\partial \phi}$). Total differentiating (37):

$$\begin{aligned}
d &= (1 - \phi)U'(\Pi^{nd}) + \phi U'(\Pi^d)(1 - f) \\
\frac{\partial S}{\partial \phi} &= -\frac{d_\phi}{d_S} \\
d_\phi &= -U'(\Pi^{nd}) + U'(\Pi^d)(1 - f) \\
d_S &= D \\
\frac{\partial S}{\partial \phi} &= -\frac{U'(\Pi^{nd}) - U'(\Pi^d)(1 - f)}{D} \leq 0
\end{aligned}$$

Equation (31 , $\frac{\partial S}{\partial f}$). Total differentiating (37):

$$\begin{aligned}
d &= (1 - \phi)U'(\Pi^{nd}) + \phi U'(\Pi^d)(1 - f) \\
\frac{\partial S}{\partial f} &= -\frac{d_f}{d_S} \\
d_f &= -U'(\Pi^d) \\
d_S &= D \\
\frac{\partial S}{\partial f} &= \frac{U'(\Pi^d)}{D} \leq 0
\end{aligned}$$

Equation (32). First Order Condition:

$$\begin{aligned}
\frac{\partial E(U)}{\partial \alpha} &= (1 - \phi)U'(\Pi^{nd})\frac{\partial \Pi^{nd}}{\partial \alpha} + \phi U'(\Pi^d)\frac{\partial \Pi^d}{\partial \alpha} \leq 0 \\
&= (1 - \phi)U'(\Pi^{nd})(-Rt) + \phi U'(\Pi^d)(-Rt + Rtf) \leq 0 \\
&= (1 - \phi)U'(\Pi^{nd})(-Rt) + \phi U'(\Pi^d)(1 - f)(-Rt) \leq 0 \\
&= (1 - \phi)U'(\Pi^{nd})(-Rt) + \phi U'(\Pi^d)(1 - f)(-Rt) \leq 0 \\
&= (1 - \phi)U'(\Pi^{nd}) + \phi U'(\Pi^d)(1 - f) \leq 0 \tag{38} \\
\frac{(1 - \phi)}{\phi(f - 1)} &\geq \frac{U'(\Pi^d)}{U'(\Pi^{nd})}
\end{aligned}$$

Equation (33). Second order condition:

$$\begin{aligned}
\frac{\partial E(U)}{\partial \alpha} &= (1 - \phi)U'(\Pi^{nd})(-Rt) + \phi U'(\Pi^d)(1 - f)(-Rt) \\
\frac{\partial^2 E(U)}{\partial \alpha^2} &= (1 - \phi)U''(\Pi^{nd})(-Rt)^2 + \phi U''(\Pi^d)(1 - f)^2(-Rt)^2 \leq 0
\end{aligned}$$

Equation (34 , $\frac{\partial \alpha}{\partial t}$). Total differentiating (38)

$$\begin{aligned}
d &= (1 - \phi)U'(\Pi^{nd})(-Rt) + \phi U'(\Pi^d)(1 - f)(-Rt) \\
\frac{\partial \alpha}{\partial t} &= -\frac{d_t}{d_\alpha} \\
d_t &= (1 - \phi)U''(\Pi^{nd})(-R\alpha)(-Rt) + \phi U''(\Pi^d)(1 - f)(-Rt)(-R\alpha - R(1 - \alpha)f) \\
d_\alpha &= D \\
\frac{\partial \alpha}{\partial t} &= -\frac{(1 - \phi)U''(\Pi^{nd})(-R\alpha)(-Rt) + \phi U''(\Pi^d)(1 - f)(-Rt)(-R\alpha - R(1 - \alpha)f)}{D} \\
\frac{\partial \alpha}{\partial t} &= \frac{R}{D} [\phi(1 - f)U''(\Pi^d)(-Rt)(\alpha + (1 - \alpha)f) + (1 - \phi)U''(\Pi^{nd})\alpha(-Rt)]
\end{aligned}$$

Where D is the second order condition. We know from (38) that:

$$(1 - \phi) = -\phi(1 - f) \frac{U'(\Pi^d)}{U'(\Pi^{nd})}$$

Replacing:

$$\begin{aligned}
\frac{\partial \alpha}{\partial t} &= \frac{R}{D} \left[\phi(1 - f)U''(\Pi^d)(\alpha + (1 - \alpha)f)(-Rt) - \phi(1 - f) \frac{U'(\Pi^d)}{U'(\Pi^{nd})} U''(\Pi^{nd})(-Rt) \right] \\
\frac{\partial \alpha}{\partial t} &= -\frac{R^2 t}{D} \phi(1 - f)U'(\Pi^d) \left[\frac{U''(\Pi^d)}{U'(\Pi^d)}(\alpha + (1 - \alpha)f) - \frac{U'(\Pi^{nd})}{U'(\Pi^d)} \right] \\
\frac{\partial \alpha}{\partial t} &= -\frac{R^2 t}{D} \phi(1 - f)U'(\Pi^{nd}) [R_A(\Pi^{nd})(\alpha + (1 - \alpha)f) - R_A(\Pi^d)] \leq 0
\end{aligned}$$

Equation (35 , $\frac{\partial \alpha}{\partial \phi}$). Total differentiating (38)

$$\begin{aligned}
d &= (1 - \phi)U'(\Pi^{nd})(-Rt) + \phi U'(\Pi^d)(1 - f)(-Rt) \\
\frac{\partial \alpha}{\partial \phi} &= -\frac{d_\phi}{d_\alpha} \\
d_\phi &= -U'(\Pi^{nd})(-Rt) + U'(\Pi^d)(1 - f)(-Rt) \\
d_\alpha &= D \\
\frac{\partial \alpha}{\partial \phi} &= -\frac{-U'(\Pi^{nd})(-Rt) + U'(\Pi^d)(1 - f)(-Rt)}{D} \\
\frac{\partial \alpha}{\partial \phi} &= -\frac{Rt}{D} [U'(\Pi^{nd}) - U'(\Pi^d)(1 - f)] \geq 0
\end{aligned}$$

Equation (36 , $\frac{\partial \alpha}{\partial f}$). Total differentiating (38)

$$\begin{aligned}d &= (1 - \phi)U'(\Pi^{nd})(-Rt) + \phi U'(\Pi^d)(1 - f)(-Rt) \\ \frac{\partial \alpha}{\partial f} &= -\frac{d_f}{d_\alpha} \\ d_f &= U'(\Pi^d)Rt \\ d_\alpha &= D \\ \frac{\partial \alpha}{\partial f} &= -\frac{U'(\Pi^d)Rt}{D} \geq 0\end{aligned}$$