An Alternative Method to Obtain the Blanchard and Kahn Solutions of Rational Expectations Models

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Abstract

In this paper we show that the method of variation of parameters can be used as an alternative procedure for solving large-scale Rational Expectations Models. Considering the same dynamical system as Blanchard and Kahn (1980) we explain, in particular, the following issues: (i) how to apply the method of variation of parameters to obtain the particular solution of the system (ii) that the solutions under the Rational Expectations or the Perfect Foresight hypothesis can be obtained after considering a boundary value problem and (iii) that some conditions must be satisfied for these solutions to be finite or bounded. We also include a brief comparison between the Rational Expectations and the Perfect Foresight solutions, where we show that the existence of a time-varying information set affects the form in which each problem is solved and interpreted. For robustness, we show that this method yields the solutions proposed in Obstfeld and Rogoff (1996), based in either recursive substitution or factorization of polynomials (considering the lag and forward operators \( L \) and \( L^{-1} \), respectively).

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1 Introduction

The objective of this note is to present the method of variation of parameters\(^1\) (MVP) as an alternative procedure to obtain the solutions of Blanchard and Kahn (1980) for solving models under the Rational Expectations or the Perfect Foresight hypothesis. By applying this method we are able to obtain the solution of the system of linear difference equations without relying in "recursive substitutions" as it is done by these authors. This aspect of the method is particularly useful, since it facilitates the computations needed to obtain the final solution of the problem.

We emphasize in this note the fact that the system under the Rational Expectations or the Perfect Foresight hypothesis can be solved as a boundary value problem. We also make explicit the conditions under which the "fundamental solution" is finite or bounded. Although mentioned in Blanchard and Kahn, these conditions are not explicitly applied in their paper.

For robustness, we compare the solutions obtained through this method with those given by Obstfeld and Rogoff (1996); that involve either recursive substitutions or factorization of polynomials. We show that all these methods yield the same results.

The rest of the paper is organized as follows. Section 2 explains the method of variation of parameters. Section 3 applies this method to the problem considered in Blanchard and Kahn (1980), which originally assumes Rational Expectations. We then illustrate this method with a simple stochastic version of the well-known Dornbusch (1976) overshooting model. Section 4 considers a natural extension: the Perfect Foresight problem. Section 5 discusses the differences in the solutions under the two behavioral assumptions and Section 6 presents concluding remarks.

2 The method

In this section we develop the method of variation of parameters. This method is widely used to solve systems of differential or difference equations in many math textbooks that deal with dynamic systems (see for differential equations Nagel et al (2004) and for difference equations Aiub (1985) or Elaydi (2005) to name just a few references). Its application to solve dynamic economic models is much less extended. One of the few examples where this method is applied is given in Turnovsky (2000).

To facilitate comparability with those solutions obtained in Blanchard and Kahn we will

\(^1\)The method of variation of parameters (also known as the method of variation of constants) was invented by J. Lagrange in 1774.
follow the notation used in their paper. Let us consider the following system of linear difference equations with constant coefficients (with a slight abuse of notation regarding the subindex $t$):

$$X_{t+1} = AX_t + \gamma Z_t, \ t \in \{t_0, t_0 + 1, \ldots, t\},$$

(1)

where $X_t$ is a $(n \times 1)$ vector of endogenous variables, $A$ is a $(n \times n)$ matrix of coefficients, $\gamma$ is a $(n \times k)$ matrix of coefficients and $Z_t$ is a $(k \times 1)$ matrix of exogenous or fundamental variables. From here onwards we assume that $A$ is a full rank matrix. We can transform $A$ as follows,

$$A = C^{-1}JC,$$

where $J$ is the Jordan canonical form of matrix $A$. Let us introduce the following change of basis: $U_t = CX_t$. The system stated in Eq. 1 thus yields,

$$U_{t+1} = JU_t + C\gamma Z_t.$$  

(2)

The homogeneous solution can be expressed as,

$$U_{t}^{h} = J^{t-t_0}K.$$  

(3)

The method of variation of parameters consists in proposing a solution for the complete system taking as a benchmark the homogeneous solution previously obtained. Since we are interested in spanning the space of solutions for the complete system, however, we have to propose a solution which is linearly independent of Eq. 3. We thus postulate,

$$U_t = J^{t-t_0}K_t,$$

(4)

where $K_t$ is a $(n \times 1)$ vector, which is now a function of time and has to be determined. Since this solution must satisfy Eq. 2 we have,

$$J^{t+1-t_0}(K_{t+1} - K_t) = C\gamma Z_t.$$  

Let $\Delta K_t \equiv K_{t+1} - K_t$ denote the difference operator on $K_t$ between periods $t + 1$ and $t$. The above system can be written as,

\footnote{To preserve comparability with Blanchard and Kahn, $C^{-1}$ denotes a matrix with the associated eigenvectors of $A$ in its columns.}

3
$$J^{t+1-t_0} \Delta K_t = C\gamma Z_t,$$

or

$$\Delta K_t = J^{-(t+1-t_0)} C\gamma Z_t,$$

Applying the sum operator on both sides of the previous expression between periods $t_0$ and $t - 1$ gives,

$$\sum_{s=t_0}^{t-1} \Delta K_s = \sum_{s=t_0}^{t-1} J^{-(s+1-t_0)} C\gamma Z_s.$$ 

Notice that the sum and the finite difference are inverse operators. It is easy to see then that $\sum_{s=t_0}^{t-1} \Delta K_s = K_t - K_{t_0}$. The above equation, therefore, reduces to:

$$K_t = K_{t_0} + \sum_{s=t_0}^{t-1} J^{-(s+1-t_0)} C\gamma Z_s,$$

expression that determines the solution of the vector $K_t$. From Eq. 4 the solution of the transformed system takes the form,

$$U_t = J^{t-t_0} K_{t_0} + \sum_{s=t_0}^{t-1} J^{(t-s-1)} C\gamma Z_s.$$ 

Considering again the transformation $U_t = CX_t$ we can obtain the solution of the original system:

$$X_t = C^{-1} J^{t-t_0} K_{t_0} + \sum_{s=t_0}^{t-1} C^{-1} J^{(t-s-1)} C\gamma Z_s.$$ 

Eq. 5 is the general solution of the system of difference equations stated in Eq. 1. If the problem is one of initial values where $X_{t_0} = X_0$ is given, then the particular solution of the problem takes the form:

$$X_t = C^{-1} J^{t-t_0} C X_0 + \sum_{s=t_0}^{t-1} C^{-1} J^{(t-s-1)} C\gamma Z_s.$$ 

---

3Note that since $A$ is a full rank matrix (i.e., all eigenvalues are different from zero), $J^{-1}$ always exists.

4This expression is also known as "telescopic sum".

5For the sum $\sum_{s=t_0}^{t_0-1} C^{-1} J^{-(s+1-t_0)} C\gamma Z_s$ we adopt the convention that is the sum of elements of an empty set. Therefore, we take the neutral value for the sum operator ($\equiv 0$). Also notice that we adopt $t_0$ as the initial period, so as to have the possibility of studying the behavior of the solution whenever $t_0$ is set to any arbitrary large value in the past.
The solution can also be expressed in terms of the original matrix $A$:

$$X_t = A^{t-t_0}X_0 + \sum_{s=t_0}^{t-1} A^{(t-s-1)}\gamma Z_s. \quad (7)$$

Note that the above solution indicates that $X_t$ is defined by the sum of the capitalized value of $X_0$ and the weighted sum of the past values of the exogenous variables.

### 3 The Rational Expectations Solution

In this section we consider an application of the method presented in the previous section under the assumption that agents have Rational Expectations (RE). We will follow, essentially, the case analysed in Blanchard and Kahn (1980). Let us define the following system of linear difference equations with constant coefficients:

$$\begin{bmatrix} X_{t+1} \\ P_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_t, \quad t \in \{t_0, t_0+1, \ldots, t, \ldots, T\}, \quad (8)$$

where $X_t$ is a $(n \times 1)$ vector of predetermined or state variables; $P_t$ is a $(m \times 1)$ vector of non-predetermined or jump variables; $Z_t$ is a $(k \times 1)$ vector of exogenous variables; $A$ is a $((n+m) \times (n+m))$ matrix of coefficients and $\gamma$ is a $((n+m) \times k)$ matrix of parameters associated with the exogenous variables of the system. Following Muth (1961), the agent’s expectation of $P_{t+1}$ will equate what the theory would predict conditioning on the information set available at time $t$, $E(P_{t+1}/\Omega_t)$; where $E(\cdot/\Omega_t)$ denotes the expectation of any given variable conditional on $\Omega_t$.

Note that the information set available at time $t$ (i.e., $\Omega_t$) not only includes the past and current realisations of the exogenous variables, but also their associated distribution functions; that will allow agents to predict rationally the mean of the future sequence of these variables.

Since $E(\cdot/\Omega_t)$ is a linear operator we can express the system stated in Eq. 8 as,

$$\begin{bmatrix} E(X_{t+1}/\Omega_t) \\ E(P_{t+1}/\Omega_t) \end{bmatrix} = A \begin{bmatrix} E(X_t/\Omega_t) \\ E(P_t/\Omega_t) \end{bmatrix} + \gamma E(Z_t/\Omega_t), \quad (9)$$

Recalling that $E(\cdot/\Omega_t)$ of any vector of variables at period $t$ is equal to the same vector of variables, from Eq. 5 the solution of the system conditional on the information set $\Omega_t$ is given by,

$$\begin{bmatrix} X_t \\ P_t \end{bmatrix} = C^{-1}J^{t-t_0}K_{t_0} + \sum_{s=t_0}^{t-1} C^{-1}J^{(t-s-1)}C\gamma Z_s. \quad (10)$$
To solve for $K_{t_0}$ we set $t = t_0$, thus obtaining: $K_{t_0} = C \begin{bmatrix} X_{t_0} \\ P_{t_0} \end{bmatrix}$. Therefore, conditional on the information set available at period $t_0 (= \Omega_{t_0})$ Eq. 10 can be written as,

$$\begin{bmatrix} E(X_t/\Omega_{t_0}) \\ E(P_t/\Omega_{t_0}) \end{bmatrix} = C^{-1} J^{t-t_0} C \begin{bmatrix} X_{t_0} \\ P_{t_0} \end{bmatrix} + \sum_{s=t_0}^{t-1} C^{-1} J^{(t-s-1)} C \gamma E(Z_s/\Omega_{t_0}).$$

Letting $t = T$ gives,

$$\begin{bmatrix} E(X_T/\Omega_{t_0}) \\ E(P_T/\Omega_{t_0}) \end{bmatrix} = C^{-1} J^{T-t_0} C \begin{bmatrix} X_{t_0} \\ P_{t_0} \end{bmatrix} + \sum_{s=t_0}^{T-1} C^{-1} J^{(T-s-1)} C \gamma E(Z_s/\Omega_{t_0}). \tag{11}$$

In order to facilitate obtaining the solution of the system, we will partition the following matrices as indicated below:

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}_{(n \times n) \times (n \times m)}, \quad C^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}_{(n \times n) \times (m \times m)}, \quad J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}_{(n \times n) \times (m \times m)}$$

and

$$\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}_{(n \times k) \times (m \times k)}.$$

With this partition, it will be possible to decouple the system depending on the whether the roots of the matrix $A$ are inside or outside the unit circle. Moreover, to guarantee the existence and uniqueness of the RE solution we will follow Blanchard and Kahn assuming that there are $n$ roots inside and $m$ roots outside the unit circle in the matrices $J_1$ and $J_2$, respectively, ordered from the lowest to the highest absolute values$^6$.

To obtain the particular solution of the RE problem we will assume the following boundary or side conditions for every period $t \in \{t_0, t_0 + 1, \ldots, t, \ldots, T\}$:

$$X_t \text{ given,} \tag{12}$$

$$\lim_{T \to +\infty} J_2^{-1} B_{22} E(P_T/\Omega_t) = 0. \tag{13}$$

Eq. 12 indicates that at period $t$, the initial or inherited value of the vector of predetermined variables is given. Eq. 13 is often called "transversality condition" when solving problems of intertemporal optimisation. It requires that as $T \to +\infty$ the expected value of the vector of jump variables conditional on the information set $\Omega_t$, discounted back to the

$^6$For a proof of this proposition see Appendix A.
current period, is zero. This condition will imply the absence of "bubbles" as is often called in the literature.

For this particular solution to be finite we further assume for every period \( t \in \{t_0, t_0+1, \ldots, t, \ldots, T\} \) that,

\[
\lim_{T \to +\infty} J_2^{-(T-t)}B_{22}^{-1}B_{21} \sum_{s=t}^{T-1} J_1^{(T-s-1)}(C_{11}\gamma_1 + C_{12}\gamma_2)E(Z_s/\Omega_t) = 0, \tag{14}
\]

Eq. 14 requires that the expected value of the vector of exogenous or "fundamental" variables, conditional on the information set \( \Omega_t \) (i.e., \( E(Z_s/\Omega_t) \)), does not grow "too fast" as \( T \to +\infty \).

Notice that we are interested in obtaining the solutions of the endogenous variables conditioning on the information set \( \Omega_t \). Solving for \( E(P_T/\Omega_t) \) Eq. 11 gives:

\[
E(P_T/\Omega_t) = (B_{21}J_1^{T-t}C_{11} + B_{22}J_2^{T-t}C_{21})X_t + (B_{21}J_1^{T-t}C_{12} + B_{22}J_2^{T-t}C_{22})P_t \\
+ \sum_{s=t}^{T-1} \{B_{21}J_1^{(T-s-1)}(C_{11}\gamma_1 + C_{12}\gamma_2) + B_{22}J_2^{(s+1-T)}(C_{21}\gamma_1 + C_{22}\gamma_2)\}E(Z_s/\Omega_t).
\tag{15}
\]

From this equation we can obtain the solution of \( P_t \) (see Appendix B for details):

\[
P_t = -C_{22}^{-1}C_{21}X_t - C_{22}^{-1} \sum_{s=t}^{\infty} J_2^{(s+1-t)}(C_{21}\gamma_1 + C_{22}\gamma_2)E(Z_s/\Omega_t), \tag{16}
\]

expression that gives a relation between \( P_t, X_t \) and the future sequences of \( Z_s \) conditional on \( \Omega_t \). Since the path of \( P_t \) is conditional on the information set available at each period \( t \), the complete solution of the system must take this fact into account. This aspect of the RE solution is a key difference with respect to the Perfect Foresight solution, as it will be discussed in the next section.

Partitioning the matrix \( A \) as \( A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \) and substituting the expression for \( P_t \) derived in Eq. 16 into Eq. 9 yields,

\[
X_{t+1} = B_{11}J_1B_{11}^{-1}X_t - A_{12}C_{22}^{-1} \sum_{s=t}^{\infty} J_2^{(s+1-t)}(C_{21}\gamma_1 + C_{22}\gamma_2)E(Z_s/\Omega_t) + \gamma_1Z_t. \tag{17}
\]

Observe that since \( X_t \) is a predetermined variable we have that \( E(X_{t+1}/\Omega_t) = X_{t+1} \). We have also made use of the following relation: \( A_{11} - A_{12}C_{22}^{-1}C_{21} = B_{11}J_1B_{11}^{-1} \). Note that Eq. 17 represents a (non-homogeneous) system of linear difference equations in \( X_t \), that can be
solved through the method of variation of parameters. The solution of this system can be obtained from Eq. 6 and is given by the following expression\textsuperscript{7},

\[
X_t = B_{11}J_1^{t-t_0}B_{11}^{-1}X_0 + B_{11}\sum_{s=t_0}^{t-1}J_1^{(t-s-1)}B_{11}^{-1}\gamma_1Z_s
\]

\[
-B_{11}\sum_{s=t_0}^{t-1}J_1^{(t-s-1)}B_{11}^{-1}A_{12}C^{-1}_{22}\sum_{v=s}^{\infty}J_2^{(-v+1-s)}(C_21\gamma_1 + C_22\gamma_2)E(Z_v/\Omega_s).
\]

Observing that \(A_{12}C^{-1}_{22} = [B_{12} - B_{11}J_1B_{11}^{-1}J_2J_2^{-1}]J_2\), it can be seen that Eq. 18 is the same as Eq. (4) in Blanchard and Kahn (1980, p. 1308) for the case in which \(t_0 = 0\). The solution of \(P_t\) is obtained by substituting Eq. 18 in Eq. 16 and considering the fact that \(C^{-1}_{22}C_1 = -B_{21}B_{11}^{-1}:
\]

\[
P_t = B_{21}J_1^{t-t_0}B_{11}^{-1}X_0 + B_{21}\sum_{s=t_0}^{t-1}J_1^{(t-s-1)}B_{11}^{-1}\gamma_1Z_s
\]

\[
-B_{21}\sum_{s=t_0}^{t-1}J_1^{(t-s-1)}B_{11}^{-1}A_{12}C^{-1}_{22}\sum_{v=s}^{\infty}J_2^{(-v+1-s)}(C_21\gamma_1 + C_22\gamma_2)E(Z_v/\Omega_s)
\]

\[
-C^{-1}_{22}\sum_{s=t}^{\infty}J_2^{(-s+1-t)}(C_21\gamma_1 + C_22\gamma_2)E(Z_s/\Omega_t).
\]

This expression is the same as Eq. (5) in Blanchard and Kahn (1980, p. 1308) whenever \(t_0 = 0\). Having obtained the solutions of \(X_t\) and \(P_t\), it is worth giving an interpretation of why there are double sums appearing in those expressions. This is a direct consequence of the presence of a time-varying information set. Notice that the sum over \(v\) is the discounted value, from any period \(s\), of the expected path of the exogenous or forcing variables. The sum in \(s\), that goes up to \(t - 1\), gives an average over those discounted values at different points in the past (i.e., starting at a different \(s\), from \(t_0\) to \(t - 1\)). The whole expression is, therefore, a weighted average seen from \(s = t_0\) up to \(t - 1\) (i.e., in the past), of the discounted value of the expected path of the exogenous variables. This particular aspect where the past "matters" in the solution of current variables is a consequence of the fact that how was perceived the expected evolution of \(Z\), conditional on each information set, affects the value of the predetermined vector of variables in previous periods; and through it implicitly affects the vector of predetermined variables in the current period.

\textsuperscript{7}To obtain this solution we can associate the matrix \(A\) in Eq. 1 with \(B_{11}J_1B_{11}^{-1}\) and \(\gamma Z_t\) with \(-A_{12}C^{-1}_{22}\sum_{v=s}^{\infty}J_2^{(-v+1-s)}(C_21\gamma_1 + C_22\gamma_2)E(Z_v/\Omega_t) + \gamma_1Z_t\). Since we know that \(A = C^{-1}JC\) we have the following relations: \(C = B_{11}^{-1}\), \(J = J_1\) and \(C^{-1} = B_{11}\). The solution thus follows.
3.1 An example: the Exchange Rate Overshooting

To illustrate this method we consider a simplified version of the well-known Dornbusch (1976)’s model developed in Taylor (1986). It is worth noting that Taylor considers the method of undetermined coefficients to solve it. The model is summarized by the following equations:

\[ m_t - p_t = -\alpha r_t \]  
\[ r_t = E(e_{t+1}/\Omega_t) - e_t \]  
\[ p_t - p_{t-1} = \beta (e_t - p_t) \]

where \( m_t \) is the nominal quantity of money, \( p_t \) is the price level, \( r_t \) is the domestic interest rate and \( e_t \) denotes the nominal exchange rate, measured as the domestic price of foreign exchange, \( \alpha > 0 \) and \( \beta > 0 \) are associated parameters. All variables, except the interest rate, are in logs. Eq. 20 defines money market equilibrium, Eq. 21 is the UIP condition where we have assumed that \( r^* = 0 \) and Eq. 22 states that domestic inflation increases in the excess demand for domestic goods (i.e., a positive function of the real exchange rate). The system can be written as:

\[
\begin{bmatrix}
p_t \\
E(e_{t+1}/\Omega_t)
\end{bmatrix}
= (1 + \beta)^{-1} \begin{bmatrix}
1 & \beta \\
\alpha^{-1} & 1 + \beta (1 + \alpha^{-1})
\end{bmatrix}
\begin{bmatrix}
p_{t-1} \\
e_t
\end{bmatrix}
+ \begin{bmatrix}
0 \\
-\alpha^{-1}
\end{bmatrix} m_t
\]

Observe that here \( p_t \) is a predetermined variable while \( E(e_{t+1}/\Omega_t) \) is a non-predetermined or jump variable. Hence, this system has the same form of the one stated in Eq. 9. As in Taylor (1986) we assume that the nominal quantity of money follows the following autoregressive process:

\[ m_t = \sum_{i=0}^{\infty} \theta_t \varepsilon_{t-i}. \]

The boundary conditions are given in this case by:

\[ p_t \text{ given} \]
\[
\lim_{T \to +\infty} \lambda_2^{-1} b_{22}^{-1} E(e_T/\Omega_t) = 0,
\]
and, to obtain a finite fundamental solution we assume:

$$\lim_{T \to +\infty} -b_{22}^{-1}\alpha^{-1}b_{21}\lambda_2^{-1}T^{1-s-1}\sum_{s=t}^{T-1} \lambda_1^{T-s-1} E \left( m_s / \Omega_t \right) = 0.$$ 

From Eqs. 18 and 19 we have the following solutions for the price level and the exchange rate:

$$p_{t-1} = \lambda_1^{-1} p_{-1} + \beta \alpha^{-1} \left(1 + \beta\right)^{-1} \sum_{s=0}^{t-1} \lambda_1^{1-s-1} \sum_{v=s}^{+\infty} \lambda_2^{-v-1-s} E \left( m_v / \Omega_s \right),$$

$$e_t = b_{21}^{-1} \lambda_1^{-1} p_{-1} + b_{21}^{-1} \beta \alpha^{-1} \left(1 + \beta\right)^{-1} \sum_{s=0}^{t-1} \lambda_1^{1-s-1} \sum_{v=s}^{+\infty} \lambda_2^{-v-1-s} E \left( m_v / \Omega_s \right) + \alpha^{-1} \sum_{s=t}^{+\infty} \lambda_2^{-(s+1-t)} E \left( m_s / \Omega_t \right).$$

Following Taylor we assume that $p_{-1} = 0$ and that the money supply is equal to zero before a temporary and unexpected monetary shock hits the economy at $t = 0$. This shock implies that the sequence of the exogenous variable is given by $E \left( m_0 / \Omega_t \right) = \epsilon_0$ and $E \left( m_t / \Omega_t \right) = 0 \quad \forall t > 0$.

The solution for the price level for $t > 0$ takes the form:

$$p_t = \beta \alpha^{-1} \left(1 + \beta\right)^{-1} \lambda_2^{-1} \lambda_1^{t-1} \epsilon_0.$$

The solution for the exchange rate at $t = 0$ is given by:

$$e_0 = \alpha^{-1} \lambda_2^{-1} \epsilon_0.$$

For $t > 0$ its solution is:

$$e_t = b_{21}^{-1} \beta \alpha^{-1} \left(1 + \beta\right)^{-1} \lambda_1^{t-1} \lambda_2^{t-1} \epsilon_0.$$

Letting $\alpha = \beta = 1$, it can be seen that the eigenvalues of the coefficient matrix are given by: $\lambda_{1,2} = 1 \mp \sqrt{2}/2$. Hence, the matrix of eigenvectors takes the form: $C^{-1} = B = \begin{bmatrix} 1 & \sqrt{2} \\ 1 - \sqrt{2} & 1 + \sqrt{2} \end{bmatrix}$. Replacing in the above solutions yields:

$$p_t = \left(1/2\right) \left(1 + \sqrt{2}/2\right)^{-1} \left(1 - \sqrt{2}\right)^{t-1} \epsilon_0$$

$$e_0 = \left(1 + \sqrt{2}/2\right)^{-1} \epsilon_0$$

$$e_t = \left(1/2\right) \left(1 - \sqrt{2}\right) \left(1 - \sqrt{2}/2\right)^{t-1} \left(1 + \sqrt{2}/2\right)^{-1} \epsilon_0.$$

Letting $\epsilon_0 = 1$, we can construct Table 1, which summarises the results obtained here. This table coincides with that presented in Taylor (1986, p. 2024).
Table 1. Effects of a transitory and unexpected shock in $m_t$

<table>
<thead>
<tr>
<th>Period</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_t$</td>
<td>0</td>
<td>0.293</td>
<td>0.086</td>
<td>0.025</td>
<td>0.007</td>
</tr>
<tr>
<td>$e_t$</td>
<td>0</td>
<td>0.586</td>
<td>-0.121</td>
<td>-0.036</td>
<td>-0.010</td>
</tr>
</tbody>
</table>

4 The Perfect Foresight Solution

Another interesting application of this method can be considered assuming that agents have Perfect Foresight. Indeed, as it will be shown in the next section, this is a particular case of the Rational Expectations solution (which is technically easier to compute). Under this behavioral assumption the system stated in Eq. 8 takes the form,

$$\begin{bmatrix} X_{t+1} \\ P_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_t, \ t \in \{t_0, \ t_0 + 1, \ldots, t, \ldots, T\}. \quad (23)$$

The side or boundary conditions of this problem are given by:

$$X_{t_0} = X_0 \ \text{given}, \quad (24)$$

$$\lim_{T \to +\infty} J^{-(T-t_0)} B_{22}^{-1} P_T = 0, \quad (25)$$

$$\lim_{T \to +\infty} J^{-(T-t_0)} B_{22}^{-1} \sum_{s=t_0}^{T-1} B_{21} J_{1}^{-(s+1-T)} (C_{11} \gamma_1 + C_{12} \gamma_2) Z_s = 0. \quad (26)$$

From Eq. 5, the solution of the system is:

$$\begin{bmatrix} X_t \\ P_t \end{bmatrix} = C^{-1} J^{-(s-1-t_0)} K_{t_0} + \sum_{s=t_0}^{t-1} C^{-1} J^{-(s+1-t)} C \gamma Z_s. \quad (27)$$

Undertaking the partitioning of the different matrices as in the previous section and including the side conditions stated in Eqs. 24-26 yields,

$$P_{t_0} = -C_{22}^{-1} C_{21} X_0 - C_{22}^{-1} \sum_{s=t_0}^{\infty} J_2^{-(s+1-t_0)} (C_{21} \gamma_1 + C_{22} \gamma_2) Z_s, \quad (28)$$

and thus $K_{t_0} = C \left[ -C_{22}^{-1} C_{21} X_0 - C_{22}^{-1} \sum_{s=t_0}^{\infty} J_2^{-(s+1-t_0)} (C_{21} \gamma_1 + C_{22} \gamma_2) Z_s \right].$ 

Noting that $C_{11} - C_{12} C_{22}^{-1} C_{21} = B_{11}^{-1},$ that $C_{22}^{-1} = B_{22} - B_{21} B_{11}^{-1} B_{12}$ and that $C_{12} C_{22}^{-1} = -B_{11}^{-1} B_{12},$ after some manipulations gives,
\[ X_t = B_{11} J_{1}^{t-t_0} \{ B_{11}^{-1} X_0 + B_{11}^{-1} B_{12} \sum_{s=t_0}^{\infty} J_2^{-s+t_0} (C_{21} \gamma_1 + C_{22} \gamma_2) Z_s \} \]

\[ + B_{11} \sum_{s=t_0}^{t-1} J_1^{-s+t} (C_{11} \gamma_1 + C_{12} \gamma_2) Z_s - B_{12} \sum_{s=t}^{\infty} J_2^{-s+t} (C_{21} \gamma_1 + C_{22} \gamma_2) Z_s, \]

and

\[ P_t = B_{21} J_{1}^{t-t_0} \{ B_{11}^{-1} X_0 + B_{11}^{-1} B_{12} \sum_{s=t_0}^{\infty} J_2^{-s+t_0} (C_{21} \gamma_1 + C_{22} \gamma_2) Z_s \} \]

\[ + B_{21} \sum_{s=t_0}^{t-1} J_1^{-s+t} (C_{11} \gamma_1 + C_{12} \gamma_2) Z_s - B_{22} \sum_{s=t}^{\infty} J_2^{-s+t} (C_{21} \gamma_1 + C_{22} \gamma_2) Z_s. \]

These expressions are equivalent to those obtained using the method of factorization; which essentially considers solving the stable roots backwards and the unstable roots forward (see, for instance, Sargent (1987) and Obstfeld and Rogoff (1996))\(^8\).

5 Comparison between the Rational Expectations and the Perfect Foresight solutions

Observe that the difference between the Rational Expectations and the Perfect Foresight solutions is associated with the fact that in the former the information set increases over time \((\Omega_t \subseteq \Omega_{t+1})\). In the PF case, in contrast, the information set remains the same from the beginning of the problem.

We can notice this difference, in particular, observing Eq. 17, since the non-homogenous term also depends on the information set of the period. To elaborate on that, notice that after some manipulations we can express Eq. 29 as follows:

\[ X_{t+1} = B_{11} J_1 B_{11}^{-1} X_t - A_{12} C_{22}^{-1} \sum_{s=t}^{\infty} J_2^{-s+t} (C_{21} \gamma_1 + C_{22} \gamma_2) Z_s \]

\[ + (B_{11} C_{11} + B_{12} C_{21}) \gamma_1 Z_t + (B_{11} C_{12} + B_{12} C_{22}) \gamma_2 Z_t. \]

Recalling that \((B_{11} C_{11} + B_{12} C_{21}) = I_{11}\) and that \((B_{11} C_{12} + B_{12} C_{22}) = 0_{12}\) yields:

\[ X_{t+1} = B_{11} J_1 B_{11}^{-1} X_t - A_{12} C_{22}^{-1} \sum_{s=t}^{\infty} J_2^{-s+t} (C_{21} \gamma_1 + C_{22} \gamma_2) Z_s + \gamma_1 Z_t. \]

\(^8\)We discuss the equivalence of our solution with that obtained in Obstfeld and Rogoff (1996) in Appendix C.
The only difference between this equation, associated with the PF problem, and Eq. 17 is the future evolution of the fundamental or exogenous variables seen from period $t$. While in the PF case the sequence of $Z_s$ is the same regardless of the period $t$ at which the solution is computed, in the RE case the expected sequence $E(Z_s/\Omega_t)$ depends on the available information at each period $t$. This fact will also imply a different set of boundary conditions for obtaining a unique bounded solution, as discussed previously.

In particular, in the PF case, imposing $n$ conditions on the vector of predetermined variables and $m$ conditions on the vector of jump variables as stated in Eqs. 24 and 25 is thus sufficient for obtaining a unique and bounded path for the vectors $X_t \forall t = t_0 + 1, \ldots, t, \ldots, T$ and $P_t \forall t = t_0, t_0 + 1, \ldots, t, \ldots, T$.

The RE solution, in contrast, requires side conditions as stated in Eqs. 12 and 13 that are specific to each particular period $t \in \{t_0, t_0 + 1, \ldots, t, \ldots, T\}$. Putting it another way, we need a different set of boundary conditions for each particular information set $\Omega_t, t \in \{t_0, t_0 + 1, \ldots, t, \ldots, T\}$, to uniquely pin down $X_t \forall t \in t_0 + 1, \ldots, t, \ldots, T$ and $P_t \forall t \in t_0, t_0 + 1, \ldots, t, \ldots, T$.

6 Concluding remarks

In this note we have emphasized two main aspects of solving linear dynamic systems: i. That finite integration can be applied to obtain the general solution of the system and ii. That the Rational Expectations or the Perfect Foresight solutions are obtained as a result of a boundary value problem.

The first point is particularly useful when recursive substitutions become cumbersome, since the method of variation of parameters saves in computations. The second point, which has been highlighted in the literature (see, for example, Taylor (1986) and Blanchard (1983)), is not often explicitly applied when solving dynamic systems.

The application of this method therefore shows that: i. It is not necessary to iterate forward and backwards a dynamic system to obtain the final solution of the problem ii. The particular solutions of the homogeneous system associated with those roots greater than one in absolute value do not appear in the particular solution of the problem (it is standard to set to zero the constant of integration associated with the unstable roots) and iii. The particular solution of the problem under Rational Expectations depends on the past and the future expected behavior of the fundamental or forcing variables of the system.

9Provided that Eq. 26 is satisfied; eliminating the possibility that the exogenous variables are growing "too fast".
Other alternative methods for solving Rational Expectations models are often considered in the literature: recursive substitutions, the method of undetermined coefficients, factorization of polynomials, etc. In principle, any of these procedures must yield the same solutions; leaving to the researcher the possibility of choosing which method to apply on the basis of personal preferences or the characteristics of the specific model under study.

It is worth noting, finally, that the boundary conditions considered to obtain the solutions of the Rational Expectations and the Perfect Foresight problems define a unique convergent saddle path towards the steady state; therefore leaving aside situations that can be characterized by "bubbles", which can be of particular interest when studying specific economic problems. Determine a formal apparatus under which those cases are contemplated in the case of large dynamic models is an interesting topic for further research.
Appendix A: existence and uniqueness revisited

In this appendix we derive some of the propositions obtained in Blanchard and Kahn (1980). To facilitate the exposition we rewrite here the general solution of the system:

\[
\begin{bmatrix}
X_t \\
P_t
\end{bmatrix} = C^{-1} J^t K_0 + \sum_{s=0}^{t-1} C^{-1} J^{(t-s-1)} C \gamma Z_s
\]

It will be assumed that the number of eigenvalues with an absolute value inside the unit circle is \((n)\), which can be different from the number of predetermined variables \((n)\). Similarly, for those eigenvalues with an absolute value outside the unit circle we have \((m)\), which can be different from the number of jump variables \((m)\). The following condition is, in any case, satisfied:

\(n + m = \bar{n} + \bar{m}\). We now partition the matrices of the system as follows:

\[
C = \begin{bmatrix}
C_{11} & C_{12} \\
\bar{n} \times n & \bar{n} \times m \\
C_{21} & C_{22} \\
\bar{m} \times n & \bar{m} \times m
\end{bmatrix},
\]

\[
C^{-1} = \begin{bmatrix}
B_{11} & B_{12} \\
\bar{n} \times n & \bar{n} \times \bar{m} \\
B_{21} & B_{22} \\
\bar{m} \times n & \bar{m} \times \bar{m}
\end{bmatrix},
\]

\[
J = \begin{bmatrix}
J_1 \\
\bar{n} \times \bar{n} \\
0 \\
\bar{m} \times \bar{n}
\end{bmatrix},
\]

\[
J = \begin{bmatrix}
J_2 \\
\bar{n} \times \bar{m} \\
0 \\
\bar{m} \times \bar{m}
\end{bmatrix}
\]

and

\[
K_0 = \begin{bmatrix}
K_1 \\
\bar{n} \times 1 \\
K_2 \\
\bar{m} \times 1
\end{bmatrix}
\]

Evaluating the vector of predetermined variables at \(t = 0\) yields,

\[
X_0 = B_{11} K_1 + B_{12} K_2.
\]

Solving for the vector of jump variables at period \(T\) gives:

\[
P_T = B_{21} J_1^T K_1 + B_{22} J_2^T K_2 + \sum_{s=0}^{T-1} \left\{ B_{21} J_1^{T-s-1} (C_{11} \gamma_1 + C_{12} \gamma_2) + B_{22} J_2^{T-s-1} (C_{21} \gamma_1 + C_{22} \gamma_2) \right\} E(Z_s/\Omega_0).
\]

To solve for \(K_2\) we pre-multiply the above expression by \(J_2^T C_{22}\). We then take the limit \(T \rightarrow +\infty\) and impose the following boundary condition:

\[
\lim_{T \rightarrow +\infty} J_2^{-T} C_{22} P_T = 0,
\]

and assume:

\[
\lim_{T \rightarrow +\infty} J_2^{-T} C_{22} B_{21} \sum_{s=t}^{T-1} J_1^{(T-s-1)} (C_{11} \gamma_1 + C_{12} \gamma_2) E(Z_s/\Omega_t) = 0.
\]

\[\text{Note that in the "singular" case } \bar{m} = 0, \text{ the vector of unknowns, } K_1, \text{ has dimension } (n + m) \text{ whereas the number of equations is equal to } n. \text{ Hence there are infinite solutions.}\]
The result is then:

\[ K_2 = - \sum_{s=0}^{+\infty} J_2^{-s-1} (C_{21}\gamma_1 + C_{22}\gamma_2) E(Z_s/\Omega_0). \]

Introducing this expression in the vector \( X_0 \) yields,

\[ X_0 = B_{11}K_1 - B_{12} \sum_{s=0}^{+\infty} J_2^{-s-1} (C_{21}\gamma_1 + C_{22}\gamma_2) E(Z_s/\Omega_0). \]

This is a system of \( n \) equations (the dimension of the vector \( X_0 \)) and \( \bar{n} \) unknowns (the dimension of the vector \( K_1 \)). Therefore, if \( n > \bar{n} \) the system is overdetermined and there are no constants \( K_1 \) that satisfy it. In contrast, if \( n < \bar{n} \) the system has more unknowns than equations and therefore there are infinite values of \( K_1 \) that satisfy it. If \( n = \bar{n} \) there exist a unique solution. Summarising, when the number of "jump" variables is lower than the number of eigenvalues inside the unit circle there are no solution to the problem. On the other hand, when the number of "jump" variables is larger than the number of eigenvalues outside the unit circle there are infinite solutions.

Appendix B

Inverting the matrix that pre-multiplies the vector \( P_t \) in Eq. 15, \( (B_{21}J_1^{T-t}C_{12}+B_{22}J_2^{T-t}C_{22})^{-1} \), taking the limit \( T \to +\infty \) having in mind that the limit of the sum (product) of functions is the sum of the limits of the functions (products), that \( J_1 \) is a matrix with the eigenvalues of \( A \) lower than one in absolute value and the properties of the inverse of a multiplication of matrices gives:

\[
\lim_{T \to +\infty} C_{22}^{-1} J_2^{-(T-t)} B_{22}^{-1} E(P_T/\Omega_t) = P_t + \lim_{T \to +\infty} C_{22}^{-1} C_{21} X_t \\
+ \lim_{T \to +\infty} C_{22}^{-1} J_2^{-(T-t)} B_{22}^{-1} \sum_{s=t}^{T-1} \{B_{21}J_1^{(T-s-1)}(C_{11}\gamma_1 + C_{12}\gamma_2) \\
+ B_{22}J_2^{-(s+1-T)}(C_{21}\gamma_1 + C_{22}\gamma_2)\} E(Z_s/\Omega_t).
\]

Applying the boundary conditions stated in Eqs. 13 and 14 yields,

\[ P_t = -C_{22}^{-1} C_{21} X_t - C_{22}^{-1} \sum_{s=t}^{+\infty} J_2^{-(s+1-t)}(C_{21}\gamma_1 + C_{22}\gamma_2) E(Z_s/\Omega_t), \]

as stated in Eq. 16.
Appendix C: comparison with the solution obtained using lag and forward operators

In this Appendix we compare our solutions with those obtained in Obstfeld and Rogoff (1996). In their Supplement C to Chapter 2 Obstfeld and Rogoff develop a method, using lag and forward operators ($L$ and $L^{-1}$ respectively), for solving a system of linear difference equations with: two variables, one predetermined and one non-predicted; one root inside and one root outside the unit circle and the assumption of Perfect Foresight.

To compare their solutions with those obtained here, we rewrite Eqs. 29 and 30 for this particular case as follows:

\[ x_t = \lambda_1^{t-t_0} x_0 + \lambda_1^{t-t_0} b_{12} \sum_{s=t_0}^{+\infty} \lambda_2^{-(s+1-t_0)} (c_{21} \gamma_1 + c_{22} \gamma_2) z_s + b_{11} \sum_{s=t_0}^{t-1} \lambda_2^{t-s-1} (c_{11} \gamma_1 + c_{12} \gamma_2) z_s - b_{12} \sum_{s=t}^{+\infty} \lambda_2^{t-s-1} (c_{21} \gamma_1 + c_{22} \gamma_2) z_s, \]

\[ p_t = b_{21} \lambda_1^{t-t_0} b_{11} x_0 + b_{21} \lambda_1^{t-t_0} b_{11} b_{22} \sum_{s=t_0}^{+\infty} \lambda_2^{-(s+1-t_0)} (c_{21} \gamma_1 + c_{22} \gamma_2) z_s + b_{21} \sum_{s=t_0}^{t-1} \lambda_1^{t-s-1} (c_{11} \gamma_1 + c_{12} \gamma_2) z_s - b_{22} \sum_{s=t}^{+\infty} \lambda_2^{t-s-1} (c_{21} \gamma_1 + c_{22} \gamma_2) z_s. \]

As in Obstfeld and Rogoff we let $t_0 \to -\infty$ and we define the "moving steady-state" values as follows:

\[ \lim_{t_0 \to -\infty} x_t = b_{11} \sum_{s=-\infty}^{t-1} \lambda_1^{t-s-1} (c_{11} \gamma_1 + c_{12} \gamma_2) z_s - b_{12} \sum_{s=t}^{+\infty} \lambda_2^{t-s-1} (c_{21} \gamma_1 + c_{22} \gamma_2) z_s \equiv \bar{x}_t \quad (31) \]

\[ \lim_{t_0 \to -\infty} p_t = b_{21} \sum_{s=-\infty}^{t-1} \lambda_1^{t-s-1} (c_{11} \gamma_1 + c_{12} \gamma_2) z_s - b_{22} \sum_{s=t}^{+\infty} \lambda_2^{t-s-1} (c_{21} \gamma_1 + c_{22} \gamma_2) z_s \equiv \bar{p}_t. \quad (32) \]

where, in order to obtain bounded solutions, we have assumed that:

\[ \lim_{t_0 \to -\infty} \lambda_1^{t-t_0} b_{12} \sum_{s=t_0}^{+\infty} \lambda_2^{-(s+1-t_0)} (c_{21} \gamma_1 + c_{22} \gamma_2) z_s = 0 \]

and

\[ \lim_{t_0 \to -\infty} b_{21} \lambda_1^{t-t_0} b_{11} b_{22} \sum_{s=t_0}^{+\infty} \lambda_2^{-(s+1-t_0)} (c_{21} \gamma_1 + c_{22} \gamma_2) z_s = 0. \]
The previous solutions can thus be expressed as:

\[ x_t = \lambda_1^{t-t_0} x_0 + b_{11} \sum_{s=-\infty}^{t_0-1} \lambda_1^{t-s-1} (c_{11} \gamma_1 + c_{12} \gamma_2) z_s - b_{12} \sum_{s=t}^{+\infty} \lambda_1^{t-s-1} (c_{21} \gamma_1 + c_{22} \gamma_2) z_s \]

\[ -b_{11} \sum_{s=-\infty}^{t_0-1} \lambda_1^{t-s-1} (c_{11} \gamma_1 + c_{12} \gamma_2) z_s + \lambda_1^{t-t_0} b_{12} \sum_{s=t_0}^{+\infty} \lambda_2^{(s+1-t_0)} (c_{21} \gamma_1 + c_{22} \gamma_2) z_s, \]

and

\[ p_t = \frac{b_{21}}{b_{11}} \lambda_1^{t-t_0} x_0 + b_{21} \sum_{s=-\infty}^{t_0-1} \lambda_1^{t-s-1} (c_{11} \gamma_1 + c_{12} \gamma_2) z_s - b_{22} \sum_{s=t}^{+\infty} \lambda_2^{t-s-1} (c_{21} \gamma_1 + c_{22} \gamma_2) z_s \]

\[ -b_{21} \sum_{s=-\infty}^{t_0-1} \lambda_1^{t-s-1} (c_{11} \gamma_1 + c_{12} \gamma_2) z_s + \frac{b_{21} b_{12}}{b_{11}} \lambda_1^{t-t_0} \sum_{s=t_0}^{+\infty} \lambda_2^{(s+1-t_0)} (c_{21} \gamma_1 + c_{22} \gamma_2) z_s, \]

We can rewrite more compactly these solutions using the definitions stated in Eqs. 31 and 32:

\[ x_t = \lambda_1^{t-t_0} (x_0 - \bar{x}_{t_0}) + \bar{x}_t, \]

and

\[ p_t = (b_{21}/b_{11}) \lambda_1^{t-t_0} (x_0 - \bar{x}_{t_0}) + \bar{p}_t, \]

expressions that are equivalent to Eq. (21) in Obstfeld and Rogoff (1996, p. 738).

References


