Abstract
This paper shows that under some general assumptions on market segmentation, general equilibrium solutions are feasible in an exchange economy where monetary shocks follow a Markovian stochastic process. Resorting to a modified version of a segmented market model by Alvarez, Atkeson and Kehoe (2002) for an exchange economy, it is shown that zero-interest rates cases are feasible, and that a close-to-liquidity trap solution emerges when agents perceive negative demand shocks as permanent. Furthermore, in contrary sense to such approach, it is also shown that it is optimal to hold cash between periods when the expected inflation rate is negative, leading to a significant persistent of these results.

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1. Introduction
The fear of inflation has biased central bank practices to privilege price stability as the utmost goal of monetary policy. The appeal to open market operations has been the most usual instrument in these cases, renewing the research on the effects of monetary policy on nominal and real variables. Last 25 years have witnessed a worldwide decline of annual inflation and interest rates. Several countries have relied on central bank practices focused on low inflation targets, and in some of them, some monetary easing was reflected in progressively lower short term rates.

In case of deliberate lowering of interest rates to avoid weakening real output growth, under a satisfactory price inflation stance, the permanent fall of interest rates to close-to-zero levels might leave monetary authorities with no margin for further stimulus. It was the recession suffered by Japan in the 1990s, the real life event that turned on some “early warnings” on the risk of close-to-zero interest rates. The observed bond yields are usually affected by the open market activities of central banks, and it can receive a different interpretation according for the alternative yield curve theories. When the Federal Reserve does it, might prefer buying short-term Treasury Bills to affect banks’ liquidity, and short-term rates, or long-term Treasury Bonds, in order to influence more directly on long-term interest rates, and on capital investment.

Laurens (2005) observed that most of central banks instrument monetary policy through open market operations, without giving up to reserve requirements and liquidity asset ratios. Furthermore, they have moved from simple rule arrangements like exchange rate pegs or monetary targeting, toward others based on the monitoring of a set of indicators, like inflation targeting, or a sort of “Taylor rules”, so that open market activities were the instrument to achieve targeted interest rates. The U.S. Federal Open Market Committee, for example, sets an interest rate target in periodical meetings, and then performs open market transactions intended to keep the overnight rate close to a targeted Fed Funds Rate. By this way, the Fed influences on commercial banks’ liquidity and the discount borrowing rate for banks overdraft positions related to reserve requirements. However, it is not the only channel that transmits money impulses to real output:

![Stylized monetary transmission channels (based on Laurens (2005))](image-url)
With regards to the interest rate channel, the term structure of the yields to maturity (i.e., baa cross section of interest rates) has become the visible set of indicators observed by everyday market players and central banks in capital markets. In case of an investor with a time horizon of four years, he/she might either a) buy and hold a four year bond, or b) buy longer maturity bonds and sell them after four years, or c) buy shorter maturity bonds and roll the mover. The so-called “expectations hypothesis of interest rates” states that under full arbitrage in complete, frictionless markets, the expected term premia between alternative investment strategies for the same maturities become zero. However, given that the “term structure” of interest rates is a cross section of interest rates that are never fixed, such hypothesis is true only when the strategies b) and c) are not correlated with aggregate risk, i.e. when the covariance between adjacent yields to maturity are not correlated (zero serial correlation). Standard knowledge on this point remarks that such a case is valid when bond traders are risk-neutral, or when there is no aggregate risk.

As those assumptions are not observed, present and lagged effects of monetary injections have a non-negligible effect on interest rates, as it appears in this figure:

![Diagram of interest rates transmission channel]

Figure 2. The interest rates transmission channel.

The modelling of the impact monetary policy on interest rates and real output (i.e., the “liquidity effect”) were addressed under three main type of models: a) setups with cash-in-advance constraints that limit the amount of money available to be invested in security markets or loans, b) models with transaction costs across markets, and c) models with specific “imperfections” or frictions, that imply deviations from the “mainstream” general equilibrium approach: heterogeneous agents, uncertainty, non-common beliefs, specific risk aversion assumptions, undiversified idiosyncratic risk. However, such mechanism is not only underlying to most of Keynesians and New Keynesian literature, but also to the oral tradition and practices in fixed income markets across the world. Its relevance with regards to other effects varies according to specifics related to inflation expectations (i.e. the “Fischer effect”) and the relevance of other monetary transmission mechanisms.

If short-term nominal rates of interest are zero, open market operations are ineffective and cannot influence on optimal consumption paths. In such circumstance, a drop long term rates is unfeasible because of bearish expectations in asset markets (agents foresee higher long term average rates), or because agents expect that nominal short-term rates would get higher, even when they are high in real terms. Under price deflation, saving in currency and government holdings are more attractive than investing in real asset prices if consumption is going down. However, it should be needed to identify the specific “friction” (or its sources) that rules out the expectation hypothesis as a plausible representation of debt markets, given that current term structure of interest rates depends on both current and expected short term rates. Given an n-period time horizon, for a time-horizon t+n (n>1), the t+n-period rate at t is

\[ i_{t,n} = \frac{1}{n} \sum_{i=0}^{n-1} i_{t+i} \Rightarrow (1 + i_{t,n})^n = (1 + i_{t,1})(1 + i_{t+1,n-1})^{n-1} \]  

(1)

Taking logarithms in both sides:

\[ i_{r,t} = \frac{1}{n} i_{r,1} + \frac{n-1}{n} i_{r+1,n-1} = \frac{1}{n} i_{r,1} + \frac{n-1}{n} i_{r+1,n-1}, \text{where} \quad 1 + i_{r+1,n-1} = E_r (1 + i_{r+1,n-1}) \]
Given Equation (2) and Jensen’s inequality \( \ln E_i(1 + i_{t+1,n-1}) < E_i \ln(1 + i_{t+1,n-1}) \), and:

\[
i_{t,n} = \frac{1}{n} [i_{t,1} + (n - 1)E(i_{t+1,n-1}) + \varepsilon_t]
\]

Long term interest rates depend on spot and future short term rates, and because of Equation (2) zero short term spot rates are not incompatible with expected positive long term interest rates, therefore, Equation (5) becomes:

\[
i_{t,n} = \frac{n-1}{n} E(i_{t+1,n-1}) + \varepsilon_t
\]

Equation (6) shows, clearly, that it is feasible, in presence of negative shocks \( \varepsilon_t < 0 \).

Recent literature, mostly inspired in the “Japan case”, have considered the zero-short term rates as the sole feature that defines a liquidity trap. Best know papers of modern literature on liquidity traps (Krugman (1998), McCallum (2000), Benhabib, Schmitt-Grohé and Uribe (2002), Eggertsson and Woodford (2003 a and b, and 2004), and Svensson (2001, 2003, 2004 a and 2004 b)), present setups that rely on credible expansionary policies as the viable tool to escape from a liquidity trap. In all them the zero-rate bound appears as the general (or maybe the only) liquidity trap case, where T-Bills market could be in equilibrium at negative (but unfeasible) nominal interest rates. Such a trap might emerge if real interest rates are relatively “high” when the nominal short rate is zero (i.e., when there is observed price deflation). However, even when price deflation plus close-to-zero interest rates might suggest a liquidity trap scenario, it should not necessarily be like that. Furthermore, not every price deflation results from an aggregate demand fall. Anyway, modern literature (mainly focused on the zero-interest rate case) has not provided definitive “guidelines” on how to avoid or escape from a liquidity trap.

This paper resorts to a modified version of the segmented model of Alvarez Atkeson and Kehoe (2002) to show how observed and expected deflation can lead an economy to zero interest rates equilibria in an exchange economy. This kind of setup shows that when a sub-set of households is excluded from financial markets, changes of monetary policy affect the distribution of real balances and of consumption across agents. There, higher interest because of negative monetary shocks lead traders to lower cash holdings, purchase more bonds, and to reduce the purchases of consumption goods. Expected bond returns go up because of the higher marginal utility of trader’s consumption. Transitory monetary expansions lead to the opposite effect, but in case of become persistent, and induce inflation expectations, long term rates might reflect in the mid-long term a sort of “Fischer effect”. In case of full participation of agents, real returns turn dependent mainly on the aggregate endowment, so monetary policy effects are more indirect, and not so unambiguous.

With an adapted version of that model (AAK, henceforth), it is shown that persistently lower interest rates may end in a liquidity trap scenario that can persist along time. Some issues related to non-neutrality of money emerge naturally given the changes applied to the model, but it shows clearly that low inflation-interest rates goals can become a dangerous strategy under not so unrealistic assumptions, especially when expected price inflation turn into well-rooted price deflation expectations. The bad news, is that it appears as needed a generally accepted theory, with solid microfoundations, that be useful to design credible contingent plans when spot and expected interest rates get close to zero.

2. The model
2.a. General framework
This section develops an intertemporal model with two segments of agents: those who actually trade in asset markets, and those who do not. In non-developed countries, segmented market models do not look as unrealistic at all, and evidence from industrial countries looks like backing the explicit introduction of market segmentation.8 In particular, constrained participation models have been successful in generating endogenous market segmentation, “liquidity effects” (a short term fall of interest rates after monetary expansion), and persistence effects from monetary shocks. 9
It will be assumed an exchange economy with homogeneous output and production technology, under the way of Lucas’ island. Each island is populated by 2-people households, so that one of them is a “worker” and the other a “shopper”. There is a finite continuum of households of measure 1. All they are living in an archipelago so that there is only one household per island, and each of them owns a tree and that produces a periodic stream of a non-storable, perishable “fruit” or “dividend” per period \( y_t \). Neither trees nor household members are mobile across islands. Production evolves with an exogenous, homogeneous and constant technology, without capital accumulation.

There is no storage technology, so the fruit cannot be a store of value. In every period the sum of fruits provides the aggregate supply for this economy and is priced in free markets. Demand for “fruits” is non-negative, and every household consumes \( (c_t) \) a fruit basket produced by the others. Its price will reflect the switches of demand and supply. For simplicity it will be assumed that substitution effects and preferences are stable, and supply random shocks will be the only source of changes in available “fruits” and in their prices. Therefore, dividends change only because of random shocks \( \xi_t \), i.e. \( y_t = y_t(\xi_t) \). Random shocks are governed by a Markov process and households do not consume their own “fruit”: they only consume others’ households fruits, and get money to buy them selling their own fruit for \( p_y \). In every period, “workers” sell their endowment to other households and receive cash in exchange. Those receipts are used by the “shopper” to purchase goods to other households and securities. Figure 1 shows the sequence of transactions, where \( q_t \) is the price of stock shares \( s_t \), \( n_t \) the price of bond securities \( B_t \), \( M_t \) are nominal balances:

\[
\begin{align*}
\text{Goods market} & \quad \text{Asset market} \\
\text{Goods market} & \quad \text{Asset market}
\end{align*}
\]

\[\xi_t, \mu_t, c_t, \Delta b_t, \Delta s_t, \Delta M_t/P_t, \xi_{t+1}, \mu_{t+1}, c_{t+1}, \Delta b_{t+1}, \Delta s_{t+1}, \Delta M_{t+1}/P_{t+1}\]

\text{Figure 3: Timing of trades}

Portfolio decisions depend upon two types of shocks: private sector idiosyncratic shocks on dividends, and monetary shocks under the way of open market operations. Households wish to maximize welfare, but only some of them participate in the asset market. Disposable income varies across agents because of idiosyncratic exogenous shocks \( \xi_n \), the impact of transfers and taxes, and because of asset returns. Without migration between islands, households exchange goods and assets every day. Goods and asset markets open and close every day. Trading agents are the members of consumer-producer households with rational expectations, facing an inter-temporal budget constraint and a cash-in-advance constraint. Here asset markets redistribute purchasing power over time. It will be assumed, for simplicity, that there credit transactions between individuals are not feasible, because of the lack of proper collaterals. Furthermore, there is a benevolent government that issues money and debt securities, facing a budget constraint and a binding transversality condition, and is assumed as “not tempted” by default incentives and monetary surprises are ruled-out. Stocks shares are issued by households, and assumed as riskier than bonds, because of the government’s ability to issue fiat money to cancel debt services.

Households can be either “rich” or “poor”. “Rich” households are those who can trade assets. Following the AAK model, they pay a fixed trading fee \( (\gamma > 0) \) per asset transaction, so there are three types of households: \( k “poor” \) ones that are not active entering to asset trading, who are active traders, and \( k \) that are not active. Poor households face a budget constraint with no bonds or stock shares in their portfolio. First order conditions provide a consumption path depending on dividend shocks, and asset trading is feasible when transaction costs are affordable. Time horizon is infinite, and households have the same utility function, but not the same consumption basket of \( j \) goods \( (c_t) \), so for the \( i \)-nth household:

\[
U_i = \int (c_{i,t}(y_i,\xi_i))d\xi_i \quad \text{or because of monotonicity of } y_i:
\]

\[
U_i = \int (c_{i,t}(\xi_i))d\xi_i
\]

All agents maximize this function, assumed as strictly increasing, concave function, defined on the real space:

\[
\text{Max } \sum_{i=0}^{\beta} U(c_{i,((\xi_i)))d\xi_i}
\]
Real balances accumulation depends on monetary shocks, price level, and dividend ("real") shocks. Their optimal consumption path depends on dividends, but for the poorest ones consumption is feasible thanks to transfers from the government. Rich households can afford transaction costs for asset trading and stochastic pricing kernels are obtained from the first order conditions of actively trading households’ dynamic optimisation. In this setup only their expectations are relevant for asset pricing.

2.b. Arbitrage conditions under Markovian monetary shocks

Under the context of well-known Lucas’ tree model, there is a representative optimizing agent that solves \( \max E_{\tilde{z}} \sum_{t=0}^{\infty} \beta^t U(c_t(z))d\tilde{z} \) subject to an intertemporal budget constraint. Objective function \( U \) is a twice-differentiable strictly concave utility function that satisfies Inada conditions, and \( \tilde{z} \) is a random shock governed by a weakly-stationary first-order Markov process. Defining \( c_t \) as the consumption per period, \( \tau_t \), the payment of a lump-sum tax; \( q_t(\tilde{z}) \) is the spot real price of a share on an asset ("tree"); \( q_t^{(1)}(\tilde{z}_{t+1}, \tilde{z}_t) \) is the expected price at \( t \) of an unit of such stock (share on any tree) in period \( t+1 \) (contingent on \( \tilde{z}_t \)). Variable \( s_t \) is the number of stocks purchased in \( t \), entitling the owner to a proportionate share in future stream of dividends \( y_t, B_t \) is the amount of bonds, \( P_t \) is the price level; and \( n_t^{(1)}(\tilde{z}_{t+1}, \tilde{z}_t) \) is the pricing kernel at period \( t \) for bonds that were purchased at \( t+1 \). Initial agent’s wealth at \( t \), in terms of the aggregate consumption goods, is \( \beta_t \). The amount of stock shares is fixed, and stock splits are unfeasible. Transfers received from the government (\( g_t \)) are lump-sum grants per beneficiary.

Proposition

Under Assumption 1, spot price at \( t \) of a share that pays dividends subject to \( t+1 \) random shock is determined by dividend plus capital gain, priced at the 1-step ahead pricing kernel for stocks, that are, in time, arbitrated to the forward pricing kernels for debt securities.

Proof

Leaving aside cash-in advance constraint, the agent solves \( \max E_{\tilde{z}} \sum_{t=0}^{\infty} \beta^t U(c_t(z))d\tilde{z} \), subject to this budget constraint on \( t \):

\[
 c_t + \tau_t + s_t \int \frac{q_t(\tilde{z}_{t+1}, \tilde{z}_t)d\tilde{z}_{t+1}}{P_t(\tilde{z}_t, \mu_t)} + \sum_{j=1}^{s_t} \left( \int \frac{q_t^{(1)}(\tilde{z}_{t+1}, \tilde{z}_t)}{P_t(\tilde{z}_t, \mu_t)} \delta_t^{(1)}(\tilde{z}_{t+1}, \tilde{z}_t)d\tilde{z}_{t+1} \right) + \int \frac{B_t(\mu_{t+1}, \mu_t)}{P_t(\tilde{z}_t, \mu_t)} n_t^{(1)}(\mu_{t+1}, \mu_t) d\mu_{t+1} \leq \theta_t(\tilde{z}_t) \tag{6}
\]

Wealth constraint on \( t+1 \) becomes:

\[
 \theta_{t+1}(\tilde{z}_{t+1}) = s_{t+1} \left[ \frac{n_t^{(1)}(\mu_{t+1}, \mu_t)}{P_t(\tilde{z}_t, \mu_t)} y_t(\tilde{z}_t) + \left( 1 + \Delta s_t \right) \right] + \sum_{j=1}^{s_{t+1}} \int \frac{q_t^{(1)}(\tilde{z}_{t+1}, \tilde{z}_t)}{P_t(\tilde{z}_t, \mu_t)} \delta_t^{(1)}(\tilde{z}_{t+1}, \tilde{z}_t)d\tilde{z}_{t+1} + \int \frac{B_t(\mu_{t+1}, \mu_t)}{P_t(\tilde{z}_t, \mu_t)} n_t^{(1)}(\mu_{t+1}, \mu_t) d\mu_{t+1} \tag{7}
\]

The next step is to switch the left side terms of (7) to the right (6) and to multiply all terms for the product between the one-period ahead shares asset pricing and the one-period ahead pricing kernel for public debt securities \( q_t^{(1)}(\tilde{z}_{t+1}, \tilde{z}_t) \). Integrating it with respect to \( \tilde{z}_{t+1} \), and substracting from (6):

\[
 c_t + \tau_t + s_t \int \frac{q_t(\tilde{z}_{t+1}, \tilde{z}_t)d\tilde{z}_{t+1}}{P_t(\tilde{z}_t, \mu_t)} \left[ \left( 1 + \Delta s_t \right) \right] \left( \frac{n_t^{(1)}(\mu_{t+1}, \mu_t)}{P_t(\tilde{z}_t, \mu_t)} y_t(\tilde{z}_t) + \left( 1 + \Delta s_t \right) \right] + \sum_{j=1}^{s_t} \int \frac{q_t^{(1)}(\tilde{z}_{t+1}, \tilde{z}_t)}{P_t(\tilde{z}_t, \mu_t)} \delta_t^{(1)}(\tilde{z}_{t+1}, \tilde{z}_t)d\tilde{z}_{t+1} + \int \frac{B_t(\mu_{t+1}, \mu_t)}{P_t(\tilde{z}_t, \mu_t)} n_t^{(1)}(\mu_{t+1}, \mu_t) d\mu_{t+1} \leq \theta_t(\tilde{z}_t) \tag{6}
\]

Assuming \( \Delta s_t = 0 \), and given that \( \int g(v'; x', v, x)dv' = 1 \) where \( v' \) and \( x' \) are Markov processes.
consistent with increasing stock prices (forward and spot), and a downward sloping yield curve signals forward changes in market value of bond securities since \( t+1 \), i.e., the slope of the yield curve. Under Because of c') the 1-step kernel for stocks depends on the expected behavior of the present value of

\[
R = \text{real gross interest rate}
\]

When monetary shocks are Markovian, and debt securities are default-free, it is legitimate to simplify c) as to allow recursive computations, so:

\[
R_j = \int q^{(j)}(\xi, \theta(\xi)) \, d\xi
\]

As the stock price is the discounted value of future dividends (goods) and capital gains, in this model the gross real interest rate is the inverse of the current price of j-step debt securities (the state, deprives the private sector of some amount of consumption goods in order to finance public expenditure) \(^{12}\). Therefore, a), b) and c) become:

\[
c_j + \tau_j + \int R_j \theta(\xi) \, q^{(j)}(\xi, \theta(\xi)) \, d\xi \leq \theta(\xi)
\]

\[
\int q_j(\xi, \xi, \xi) \, d\xi = \left[ \int y_j(\xi, \xi) \, d\xi + \int q_{j+1}(\xi_{j+1}, \xi_{j+1}) \, d\xi_{j+1} \right] \left[ 1 + \pi_j \right]
\]

\[
R_j = \text{real gross interest rate}
\]

Because of c') the 1-step kernel for stocks depends on the expected behavior of the present value of forward changes in market value of bond securities since \( t+1 \), i.e., the slope of the yield curve. Under the so-called expectations “theory” of the term structure, an upward sloping yield curve, should be consistent with increasing stock prices (forward and spot), and a downward sloping yield curve signals decreasing forward and spot stock prices. Unless unbounded growing dividend stream be feasible, stock price cannot be infinite under zero-arbitrage condition. Furthermore, only with random shocks are persistently negative, the stock share kernel will converge in the limit-case to zero. \(^{15}\)

2.c. Optimizations

“Poor” households are “liquidity constrained” agents, with usual budget constraint and a cash-in-advance constraint. Relevant pricing kernels are obtained from the optimisation of wealthy households that are active asset traders and interest rates depend on a subset of agents: wealthy and actively trading “shoppers”. From financial and wealth constraints, and aggregating across households, it is
obtained a negative relationship between expected inflation rate and real balances demand. In case of both low demand for assets, and expected price deflation, asset trading liquidity tends to zero.

**Optimisation I: “Poor” households**

\[
\text{Max } E \sum_{i} \beta^i \int U(c_{it} (\xi_i)) d\xi_i \\
\text{subject to:}
\]

\[
\frac{M_{u}^p (\mu_i)}{P_t (\xi_i, \mu_i)} \leq \theta^p_i (\xi_i)
\]

\[
\theta^p_i (\xi_{i+1}) \leq \frac{M_{u}^p (\mu_i) + \zeta (\xi^i) \cdot (\xi^i - P_t (\xi_i, \mu_i) \cdot c_{it} (\xi_i))}{P_{i+1} (\xi_{i+1}, \mu_{i+1})}
\]

\[
\zeta (\xi^i) = \begin{cases} 
1, & \text{when identified as poor} \\
0, & \text{when are not "poor" enough}
\end{cases}
\]

\[
M_{u}^p \geq P_t (c_{it} (\xi_i))
\]

\[
\theta^p_i (\xi_0) = \frac{M_{u}^0}{P_0}
\]

Under a binding cash-in-advance constraint (expression (11) as a strict equality), fruit trade is their sole source of funds, unless they be so poor that receive a lump-sum grant from the government. Without monitoring costs for the government, the probability of being identified as a poor, is dependent on the entire history of idiosyncratic random shocks. The problem is, therefore, constrained to a typical wealth accumulation one:

\[
\text{Max } \sum_{i=0}^{\infty} \beta^i \int U(c_{it} (\xi_i)) d\xi_i \\
\text{s.t.:}
\]

\[
\frac{M_{u}^p (\mu_i)}{P_t (\xi_i, \mu_i)} \leq (y_i (\xi_i) + \xi^i \cdot g_i - c_{it} (\xi_i)) \cdot P_{i+1}
\]

The associated Bellman equation is:

\[
\text{Max } V_t = E \{ U(c_{it} (\xi_i)) d\xi_i + \beta \cdot V_{i+1} (\theta_{i+1}) \}
\]

\[
\text{s.t.: } \theta_{i+1} = \theta_i - c_{it} (\xi_i)
\]

First order conditions:

\[
U^*(c_{it} (\xi_i)) + \beta \cdot V^*(\theta_{i+1}) \cdot \theta_{i+1} (c_{it} (\xi_i)) = 0
\]

Envelope theorem:

\[
V^*(\theta_i) = \beta \cdot V^*(\theta_{i+1})
\]

Benveniste-Sheinkman condition:

\[
U^*(c_{it} (\xi_i)) = V^* (\theta_i)
\]

Euler equation:

\[
U^*(c_{it} (\xi_i)) = \beta \cdot U^*(c_{it+1} (\xi_{i+1}))
\]

**Optimisation II: Wealthy households**

These households have enough disposable income to accumulate financial assets apart of money balances. Given their objective functions, exogenous shocks, and transaction costs for asset trading, they may be actively trading assets (i.e. transferring nominal balances to bond and stock shares purchases, or selling them in exchange for nominal balances).

Therefore they might be participating as active traders (“active households”) or not (“inactive households). Asset trading is centralized in a single market where all participants pay a fixed transaction fee per operation to the only financial intermediary. For sake of simplicity, this is assumed as a single asset broker that does also charge that fee to wealthy shoppers that participate as central bank’s counterpart in open market operations. This intermediary is a strict broker, not a dealer either an underwriter, nor a commercial or investment bank or fund. Transaction costs are fixed like in the AAK model.
The associated Bellman equation is

\[ \max \{ E \sum_{i=0}^{\infty} \beta^i \left[ U \left( c_{n, t+1} \left( \xi_t, \mu_{t+1} \right) \right) d \xi_t, d \mu_{t+1} \right] \}, \text{ facing these constraints:} \]

\[ \quad \frac{M_{\xi, t} \left( \mu_{t+1} \right)}{P_{\xi, t} \left( \mu_{t+1} \right)} + \tau_t \left( \xi_t \right) + \int s_t^\gamma \left( \xi_t, q_t \left( \xi_{t+1}, \xi_t \right) \right) d \xi_{t+1} + R_t \left( \beta_t^1 \left( \mu_t \right) n_t \left( \mu_{t+1}, \mu_t \right) d \mu_{t+1} \leq \theta_t \left( \xi_{t+1}, \mu_{t+1} \right) \right. \]

\[ \left. \left( 18 \right) \right. \]

\[ \theta_t \left( \xi_{t+1}, \mu_{t+1} \right) = \frac{M_{\xi, t} \left( \mu_{t+1} \right)}{P_{\xi, t} \left( \mu_{t+1} \right)} + \tau_t \left( \xi_t \right) + \int s_t^\gamma \left( \xi_t, q_t \left( \xi_{t+1}, \xi_t \right) \right) d \xi_{t+1} \]

\[ + R_t \left( \beta_t^1 \left( \mu_t \right) n_t \left( \mu_{t+1}, \mu_t \right) d \mu_{t+1} \right) \]

\[ + \frac{M_{\xi, t} \left( \mu_{t+1} \right)}{P_{\xi, t} \left( \mu_{t+1} \right)} \cdot P_{\xi, t} \left( \mu_{t+1} \right) \cdot B_t \left( \mu_t \right) \]

\[ + \frac{M_{\xi, t} \left( \mu_{t+1} \right) - P_{\xi, t} \left( \mu_{t+1} \right) \cdot c_t^1 \left( \xi_t, \mu_{t+1}, \mu_t \right) + \tau_t \left( \xi_t \right) - R_t \left( \beta_t^1 \left( \mu_t \right) n_t \left( \mu_{t+1}, \mu_t \right) d \mu_{t+1} \right) \]

\[ = \frac{\left[ R_t \left( \beta_t^1 \left( \mu_t \right) n_t \left( \mu_{t+1}, \mu_t \right) d \mu_{t+1} + \int s_t^\gamma \left( \xi_t, q_t \left( \xi_{t+1}, \xi_t \right) \right) d \xi_{t+1} - \left( x \left( \mu_t \right) + \gamma \right) \cdot z \left( \mu_t \right) \right] P_{\xi, t} \left( \mu_{t+1} \right) \left( \xi_{t+1} \right)}{P_{\xi, t} \left( \mu_{t+1} \right) \left( \xi_{t+1} \right)} \]

\[ \left( 19 \right) \]

\[ M_{\xi, t} \left( \mu_{t+1} \right) \geq P_t \left( \xi_t, \mu_t \right) c_t^1 \left( \xi_t, \mu_{t+1}, \mu_t \right) + \left( x \left( \mu_t \right) + \gamma \right) \cdot z \left( \mu_t \right) - \tau_t \left( \xi_t \right) \]

But: \[ R_t^{-1} \int b_t^1 \left( \mu_t \right) n_t \left( \mu_{t+1}, \mu_t \right) d \mu_{t+1} + \int s_t^\gamma \left( \xi_t, q_t \left( \xi_{t+1}, \xi_t \right) \right) d \xi_{t+1} = x \left( \mu_t \right), \text{ so:} \]

\[ M_{\xi, t} \left( \mu_{t+1} \right) - P_t \left( \xi_t, \mu_t \right) \cdot \left( R_t^{-1} \int b_t^1 \left( \mu_t \right) n_t \left( \mu_{t+1}, \mu_t \right) d \mu_{t+1} + \int s_t^\gamma \left( \xi_t, q_t \left( \xi_{t+1}, \xi_t \right) \right) d \xi_{t+1} + \gamma \right) \cdot z \left( \mu_t \right) \geq \]

\[ \geq P_t \left( \xi_t, \mu_t \right) \cdot c_t^1 \left( \xi_t, \mu_{t+1}, \mu_t \right) - \tau_t \left( \xi_t \right) \]

\[ \left( 20 \right) \]

\[ z = 0, \text{ if transfer } = 0 \]
\[ 1, \text{ if transfer } \neq 0 \]

\[ \gamma > 0: \text{ transaction cost per asset trading transaction} \]
\[ x \left( \mu_t \right) \geq 0: \text{ amount of money "used" in asset trading} \]

Equivalently:

\[ P_t \left( \xi_t, \mu_t \right) c_t^1 \left( \xi_t, \mu_{t+1}, \mu_t \right) \leq M_{\xi, t} \left( \mu_{t+1} \right) - P_t \left( \xi_t, \mu_t \right) \left( R_t^{-1} \int b_t^1 \left( \mu_t \right) n_t \left( \mu_{t+1}, \mu_t \right) d \mu_{t+1} + \int s_t^\gamma \left( \xi_t, q_t \left( \xi_{t+1}, \xi_t \right) \right) d \xi_{t+1} + \gamma \right) \cdot z \left( \mu_t \right) - \tau_t \left( \xi_t \right) \]

\[ \left( 21 \right) \]

When \( t = 0 \):

\[ \theta_0 \left( \xi_0, \mu_t \right) = \frac{M_{\xi, 0} \left( \mu_{t+1} \right)}{P_{\xi, 0} \left( \mu_{t+1} \right)} + \tau_0 + \int s_0^\gamma \left( \xi_0, q_0 \left( \xi_{t+1}, \xi_0 \right) \right) d \xi_1 + R_0^{-1} \int b_0^1 \left( \mu_0 \right) n_0 \left( \mu_{t+1}, \mu_0 \right) d \mu_1 \]

\[ \left( 22 \right) \]

When \( z = 1 \), Equation (20) is the cash-in-advance constraint for "active" rich households, when \( z = 0 \) it corresponds for and for "inactive" households’ problem.

The associated Bellman equation is

\[ \text{Max} \{ E, \sum_{i=0}^{\infty} \beta^i \left[ U \left( c_{n, t+1} \left( \xi_t, \mu_{t+1} \right) \right) d \xi_t, d \mu_{t+1} \right] \} \]

\[ \text{st.} \quad \theta_t \left( \xi_{t+1}, \mu_{t+1} \right) = \theta_t \left( \xi_t, \mu_{t+1} \right) - c_t^1 \left( \xi_t, \mu_{t+1}, \mu_t \right) \]

The intertemporal allocation for "active" rich agents is solved with the Lagrangean:

\[ L = \left[ E, \sum_{i=0}^{\infty} \beta^i \left[ U \left( c_{n, t+1} \left( \xi_t, \mu_{t+1} \right) \right) d \xi_t, d \mu_{t+1} \right] \right] + \beta \left[ \sum_{t=0}^{\infty} v_t \left( \theta_t \left( \xi_{t+1}, \mu_{t+1} \right) - c_t^1 \left( \xi_t, \mu_{t+1}, \mu_t \right) - \tau_t \left( \xi_t \right) \right) \right] \]

\[ \lambda_t > 0 \]

Cash-in-advance constraint (20) is binding, and money injections are through open market operations. First order conditions (small caps indicate “real” magnitudes) are:

\[ c_t^1: \]

\[ U^1 c_t^1 \left( \xi_t, \mu_{t+1}, \mu_t \right) = \lambda_t \quad (= \lambda_t \text{ when } c_t > 0) \]

\[ \left( 23 \right) \]

\[ \left( 24 \right) \]

\[ s_{t+1}: \]

\[ \beta \cdot \frac{\partial v_{t+1}}{\partial \theta_t} \int q_{t+1} \left( \xi_{t+1}, \xi_t \right) d \xi_{t+1} = \lambda_t \int q_t \left( \xi_{t+1}, \xi_t \right) d \xi_{t+1} \]
Inserting this (23) and (29) in (24):

\[ \beta \cdot U'(c_{\xi, t}^{i}(\xi_{t+1}^{i}, \mu^{t+1}, \mu_{t+1}^{t})) \cdot \int q_{i, t}(\xi_{t+1}^{i}, \xi_{t+1}^{i})d\xi_{t+1}^{i} = U'(c_{\xi, t}^{i}(\xi_{t}^{i}, \mu^{t}, \mu_{t}^{t}))\int q_{i, t}(\xi_{t+1}^{i}, \xi_{t}^{i})d\xi_{t+1}^{i} \]

From here, and because of (27):

\[ \int q_{i, t}(\xi_{t+1}^{i}, \xi_{t}^{i})d\xi_{t+1}^{i} = \beta \cdot U'(c_{\xi, t}^{i}(\xi_{t+1}^{i}, \mu^{t+1}, \mu_{t+1}^{t})) \int q_{i, t}(\xi_{t+1}^{i}, \xi_{t}^{i})d\xi_{t+1}^{i} = R_{t}^{-1} \int q_{i, t}(\xi_{t+1}^{i}, \xi_{t}^{i})d\xi_{t+1}^{i} \]

Inserting (26) and (23) in the first order condition (25), and because of Euler equation:

\[ \beta \cdot U^{i}(c_{\xi, t}^{i}(\xi_{t+1}^{i}, \mu^{t+1}, \mu_{t+1}^{t})), \int n_{i, t}^{t}(\mu_{t+1}^{t}, \mu_{t}^{t})d\mu_{t}^{t+1} = U^{i}(c_{\xi, t}^{i}(\xi_{t}^{i}, \mu^{t}, \mu_{t}^{t})), \int n_{i, t}^{t}(\mu_{t+1}^{t}, \mu_{t}^{t})d\mu_{t}^{t+1} = R_{t}^{-1} \int n_{i, t}^{t}(\mu_{t+1}^{t}, \mu_{t}^{t})d\mu_{t}^{t+1} \]

As t+1 shocks and 1-step forward price of either bonds or stocks are the needed information to compute the dynamics of the system, given that stock and bond securities will evolve growing at the same rate. As a consequence, any negative shock for asset price markets hits both debt and equity security prices. From (31), the pricing kernels for stocks and bonds are:

\[ \int q_{i, t}(\xi_{t+1}^{i}, \xi_{t}^{i})d\xi_{t+1}^{i} = \beta \cdot U'(c_{\xi, t}^{i}(\xi_{t+1}^{i}, \mu^{t+1}, \mu_{t}^{t})), \int n_{i, t}^{t}(\mu_{t+1}^{t}, \mu_{t}^{t})d\mu_{t}^{t+1} \]

Because of recursive forward iteration:

\[ \int q_{i, t}(\xi_{t+1}^{i}, \xi_{t}^{i})d\xi_{t+1}^{i} = \beta \cdot \prod_{j=t}^{\infty} U^{j}(c_{\xi, j}^{j}(\mu_{j+1}^{j}, \mu^{j+1}, \xi_{j+1}^{j})), \int q_{j+1}(\xi_{j+1}^{j}, \xi_{j+1}^{j})d\xi_{j+1}^{j} \]

Assuming complete markets, and because of Equations (26) and (27):

\[ \int q_{i, t}(\xi_{t+1}^{i}, \xi_{t}^{i})d\xi_{t+1}^{i} = \prod_{j=t}^{\infty} R_{j+1}^{-1} \int q_{j+1}(\xi_{j+1}^{j}, \xi_{j+1}^{j})d\xi_{j+1}^{j} \]

The higher the gross real interest rate, and the lower the future price for stocks and bonds, and the lower the spot prices for both assets. The behaviour of asset prices, explicitly depend on future
monetary and real shocks. The second equation implies that the yield curve for debt securities will depend on the nature of the stochastic process for monetary shocks. Using this identity from Manuelli and Sargent (1987):

\[ R_{j,t} = \int q_j(\xi, \xi) d\xi \]

From equation (33)\(^{18}\):

\[ \int n_j(\mu_{t+1}, \mu_t) d\mu_{t+1} = \beta \frac{U^{\prime t+1}_{i,t+1}(c_{t+1}(\xi, \xi, \mu_{t+1}, \mu_{t+2}))}{U^{\prime t}_{i} (c_{t}(\xi, \xi, \mu_{t}, \mu_{t+1}))} R_{t+1} \int n_j(\mu_{t+2}, \mu_{t+1}) d\mu_{t+2} \]

The riskless bond (1-period bond), can be defined as:

\[ \int n_j(\mu_{t+1}, \mu_t) d\mu_{t+1} = R_{t+1} = \beta \frac{U^{\prime t+1}_{i,t+1}(c_{t+1}(\xi, \xi, \mu_{t+1}, \mu_{t+2}))}{U^{\prime t}_{i} (c_{t}(\xi, \xi, \mu_{t}, \mu_{t+1}))} R_{t+1} \int n_j(\mu_{t+2}, \mu_{t+1}) d\mu_{t+2} \]

Because of first order conditions (25), (27) and (28):

\[ 1 = \beta \frac{U^{\prime t+1}_{i,t+1}(c_{t+1}(\xi, \xi, \mu_{t+1}, \mu_{t+2}))}{U^{\prime t}_{i} (c_{t}(\xi, \xi, \mu_{t}, \mu_{t+1}))} R_{t+1} \]

The expectations theory of the term structure assumes that the slope of the yield curve depends on the expectations about future interest rates. The slope will be positive and rates will rise with the length of maturity in case that market participants anticipate an increase of interest rates, and vice-versa. So, the delivery price of forward contracts will reflect the markets’ expectations about future spot rates and bond prices. In this case this prediction is compatible with this inter-temporal model when successive conditional covariance terms are zero. It is well known that those covariances are zero when agents are risk neutral, the marginal rate of inter-temporal substitution is constant and equal to \( \beta \), and spot and forward prices of bonds are constants, like all the array of total rates of return. In such case, and assuming independent successive marginal rates of inter-temporal substitution:

\[ \int n_j(\mu_{t+1}, \mu_t) d\mu_{t+1} = \prod_{i=1}^{\infty} E_i \left[ \int n_j(\mu_{t+1}, \mu_t) d\mu_{t+1} \right] \]

Because of simple algebra:

\[ \int n_j(\mu_{t+1}, \mu_t) d\mu_{t+1} = \prod_{i=1}^{\infty} E_i \left[ \int n_j(\mu_{t+1}, \mu_t) d\mu_{t+1} \right] \]

\[ \int n_j(\mu_{t+1}, \mu_t) d\mu_{t+1} = R_{t+1} \int n_j(\mu_{t+2}, \mu_{t+1}) d\mu_{t+2} \]

\[ \int n_j(\mu_{t+1}, \mu_t) d\mu_{t+1} = \prod_{i=1}^{\infty} R_{t+1} \int n_j(\mu_{t+2}, \mu_{t+1}) d\mu_{t+2} \]

\[ \prod_{i=1}^{\infty} (R_{t+1})^{-1} = \prod_{i=1}^{\infty} n_j(\mu_{t+1}, \mu_t) d\mu_{t+1} \Rightarrow \prod_{i=1}^{\infty} n_j(\mu_{t+2}, \mu_{t+1}) d\mu_{t+2} = 1 \]

\[ R_{t+1} = \int n_j(\mu_{t+1}, \mu_t) d\mu_{t+1} \sqcup \prod_{i=1}^{\infty} (R_{t+1})^{-1} \rightarrow 1 \]

Spot interest rates will increase (decrease) only when the bond issue is higher (lower) than the amount of money to be spent in securities without changes in consumption decisions. Therefore, forward interest rates’ reaction to open market operations will depend on the impact of present monetary shocks on both future interest rates and on the longest term interest rates\(^{18}\). However, such effect is not independent of the impact of all interest rates on present and expected aggregate demand. With no shocks, and generalizing for any \( t \) arbitrage between a long term bond and a series of consecutive 1-period bond securities implies: \( \prod_{j=k}^{R_{t+1}} = R_{t+1} \).

For nominal interest rates \( (1 + i_{t+1})^n = \prod_{i=0}^{n-1} (1 + i_{t+1}) \). Taking logarithms in both sides:

\[ i_{t,n} = \frac{1}{n} \sum_{i=0}^{n-1} i_{t+1} \Rightarrow (1 + i_{t,n})^n = (1 + i_{t,1})(1 + i_{t+1,n-1})^{n-1} \]

and

\[ i_{t,n} = \frac{1}{n} i_{t+1} + \frac{n-1}{n} i_{t+1,n-1}, where \quad 1 + i_{t+1,n-1} = E_i(1 + i_{t+1,n-1}) \]
Forward and spot markets are directly linked thanks to the expectations of future interest rates, and an n-period bond that pays $1 in any state of the world in period t+n, in period t is worth:

$$\int n(\mu_{t+n}, \mu_t) \, d\mu_{t+n} = (1 + r_{t,n})^{-1} = \beta^n E_t U_{t+n}^k (c_{t+n}^- (\xi_{t+n}, \mu_{t+n}, \mu_{t+n+1})) P_{t+n} = R^{-1}_{t,n} P_{t+n}$$

Equation (49) is a generalized version of the pure expectations theory of the term structure of interest rates. Because of the definition of correlation coefficient between adjacent gross interest rates

$$1$$

From Equation (40):

$$\beta U_{t+1}^k (c_{t+1}^- (\xi_{t+1}, \mu_{t+1}, \mu_{t+2})) R_{t+1} = 1$$

but for n = 2:

$$\beta U_{t+n}^k (c_{t+n}^- (\xi_{t+n}, \mu_{t+n}, \mu_{t+n+1})) R_{t+n} = 1$$

Because of the “law” of iterated expectations:

$$R_{t+1} = E_t \beta U_{t+1}^k (c_{t+1}^- (\xi_{t+1}, \mu_{t+1}, \mu_{t+2})) E_t R_{t+n+1} + \text{cov}_t (\beta U_{t+1}^k (c_{t+1}^- (\xi_{t+1}, \mu_{t+1}, \mu_{t+2})), R_{t+n+1})$$

Using the definition of conditional covariance:

$$R_{t+1} = R_{t+n+1} E_t R_{t+1} + \text{cov}_t (\beta U_{t+1}^k (c_{t+1}^- (\xi_{t+1}, \mu_{t+1}, \mu_{t+2})), R_{t+n+1})$$

Because of equation (40):

$$R_{t+1} = R_{t+n+1} E_t R_{t+1} + \text{cov}_t (\beta U_{t+1}^k (c_{t+1}^- (\xi_{t+1}, \mu_{t+1}, \mu_{t+2})), R_{t+n+1})$$

Equation (49) is a generalized version of the pure expectations theory of the term structure of interest rates, adjusted for the risk premium for covariances between “adjacent” 1 step-ahead gross interest rates. Because of the definition of correlation coefficient between adjacent gross interest rates

$$R_{t+1} = R_{t+n+1} E_t R_{t+1} + \rho_t (R_{t+n+1}, R_{t+n+1}) \sqrt{Var_R_{t+n+1}}$$

For n = 3:

$$R_{t+1} = R_{t+n+1} E_t R_{t+1} + R_{t+n+1} E_t R_{t+n+2} + \rho_t (R_{t+n+1}, R_{t+n+2}) \sqrt{Var_R_{t+n+1}} \sqrt{Var_R_{t+n+2}}$$

Because of forward iteration:

$$R_{t+1} = \prod_{i=t}^{t+n} E_t \beta U_{t+i}^k (c_{t+i}^- (\xi_{t+i}, \mu_{t+i}, \mu_{t+i+1})) = \prod_{i=t}^{t+n} \beta E_t U_{t+i}^k (c_{t+i}^- (\xi_{t+i}, \mu_{t+i}, \mu_{t+i+1}))$$

$$R_{t+1} = \sum_{i=t}^{t+n} E_t R_{t+i} \text{cov}_t (R_{t+i}, R_{t+i}) \sqrt{Var_R_{t+i}} \sqrt{Var_R_{t+i}}$$

It is useful to remark that it is not assumed the observance of the segmented markets hypothesis for the term structure. In such case it is assumed that markets for bonds of different maturities are completely separated and segmented. In such case those bonds are not substitutes and term spreads should exhibit a significant persistence along time.

3. Existence of a general equilibrium

As a proportion on the whole population, this economy has k households that are poor (or “liquidity constrained”) and k wealthy or “rich” households where it can be distinguished k “rich” that are active, and k inactive. Given the optimization of section 2, this economy is in equilibrium.

**Definition:** An equilibrium is a set of:

- initial conditions $M_0(\mu_0) > 0, b_0(\mu_0) \geq 0$
- stochastic processes $\begin{bmatrix} \xi_t, \xi_i, t_i, P, k, P, k, M_{t+1}, s_k, k, b_k, h_k, h_k, t, h_k, h_k, t+1 \end{bmatrix}$
- pricing functions $q_i(\xi_{t+1}, \xi_t)$ and $n_i(\mu_{t+1}, \mu_t)$ such that:
the government chooses tax, and deficit financing strategies $\{r_t, b_{t+1}(\mu_{t+1}), M_{t+1}(\mu_{t+1})\}_{t=0}^\infty$

such that for any $t \geq 0, m_t, m_{t+1}^e, m_{t+1}^v, m_t^e = m_t^v = g_t$, and the budget constraint is satisfied:

$$g_t = r_t + R_{\xi_t}^{-1} \left[ b_{t+1}^e(\mu_{t+1}) m_{t+1}(\mu_{t+2}, \mu_{t+1}) d\mu_{t+2} = -b_t^e(\mu_{t+1}) m_t(\mu_{t+1}, \mu_t) d\mu_{t+1} + \frac{m_{t+1}(\mu_{t+1}) - M_t(\mu_{t+1})}{P(\xi_t, \mu_t)} \right]$$

1) given the above mentioned pricing functions, stochastic processes, and initial conditions, poor and rich households solve the Optimization Problems I and II, respectively, and
2) asset markets clear each period.

**Proposition i.** The equilibrium above defined implies that because of Walras’ law, goods markets are in equilibrium upon the following assumptions:

a) Given $(\xi_t, \mu_t, \mu_{t+1}) \equiv s_t \in S \subset \Re^n$, for any $n$, and $c_t \in C \subset \Re^k$ for any $k$, there is an arbitrary real number $K$ such that $|v_t(s_t, c_t)| \leq K$ for all $(s_t, c_t) \in S \times C$.

b) Utility function $U(\cdot)$ is well defined for both wealthy and poor households as an upper bounded and continuous function on $S \times C$.

c) Transition function for wealth accumulation is continuous on $S \times C$.

d) There is a feasible consumption correspondence $(c_t \equiv \phi(s_t)) : S \rightarrow P(C)$ that specifies for each $s_t \in S$ the set $\phi(s_t) \subset C$ of consumption actions available at $s_t$.

e) $\phi(s_t)$ is a continuous, compact-valued correspondence on $S$.

**Proof:** Sargent (1987), Ch. 5, and Sundaram (2002), Ch. 12.

**Proposition ii.** Under Proposition i, and two additional assumptions:

1. utility functions are strictly defined as continuous functions on the positive real field, and that they fulfil Inada conditions

2. $v_t$ is nondecreasing on $S$, and strictly increasing and concave on $\Re^+$,

there exists a stationary optimal strategy that allows to meet a well defined tuple

$$T = \left\{ \xi_t, g_t, c_t, r_t, m_t^p, m_t^p, m_t^p, m_t^p, M_t, s_t, s_t, b_t, b_t, b_t, b_t, b_t, b_t \right\}_{t=0}^\infty$$

that meets the compactness and continuity conditions that guarantee the existence of a single optimal strategy, such that satisfy Bellman equations of Optimisations I and II.

**Proof:** Sundaram (2002), Ch. 12.

**Proposition iii.** Under the assumptions of Propositions i and ii, and with concave transition functions, the value function $v_t$ is strictly concave and there is a unique optimal strategy continuous on $S$, such that Bellman Equation is single-valued on $S$, and it is a single maximum.

**Proof:** Sundaram (2002), Ch. 12.

Under the assumptions of concavity for utility and value functions, there exists a price vector that supports a general equilibrium, with the observance of these Propositions.

4. Close-to-zero interest rates and the peril of liquidity traps

Appendix B shows the arbitrage conditions for different assets and debt maturities according to the pure expectation theory. Underlying pricing kernels support an intertemporal general equilibrium. Such mechanics was explored by Lucas (1990), for 1-period discount bonds. In case of a low demand level for bonds, their prices fall below par value. The simplified version of Lucas’ approach shows that market value is moving pari-passu with the liquid funds allocated to active asset trading. When such liquidity falls to zero, bond prices do also fall to zero. The growth rate of nominal yields to maturity $(i_t)$ maintains an inverse relationship to bond prices $(z_t)$, and a direct one to the growth rate of funds flows $x_t$ to debt investments:

$$\frac{d \ln i_t}{dt} = \frac{d \ln x_t}{dt} - \frac{d \ln z_t}{dt}$$

In case of a full match between bond supply and demand (and debt prices at par value), interest rates do not change. The government might induce a fall of $i_t$ through expansionary open market activities purchasing new bonds in $t+1$ (interest rates decline).
When demand for new debt (i.e. the growth rate of liquidity devoted to such purchases) is determined by forward-looking rational expectations on open market activities, the effective change of liquidity in asset markets depends on two groups of households: those that are active traders, and those that are not, but are potentially active traders: \( t+1 \) asset market-liquidity growth will depend on the history of monetary shocks up to date \( t \).

When agents prefer to invest funds in asset trading, “liquidity effect” is a normal consequence of central bank’s trading. Either in high or low inflation, expected increase of nominal balances implies a higher utility from active trading, and greater incentives to become active traders. In the extreme, capital amortization of all outstanding bonds, with permanent monetary expansions lead agents sooner or later to flight from money to commodities and to stock shares, so no wealthy household be inactive. Therefore it results a U shaped relationship between the expected utility from asset trading and the level of real balances of wealthy households, as dependent of monetary shocks. Interest rates might decline because of high demand on public debt (an increase of activity and liquidity in asset markets allocated to public debt), or because of negative monetary shocks (negative rate of change for outstanding debt), that discourage households to allocate liquidity to asset trading. In case of permanent negative monetary shocks it results a drop in the level and in the growth rate of liquidity devoted to asset trading. In the limit asset trading falls to zero: agents do not wish to buy bonds because of expected capital losses.

Let be assumed constant present and expected dividends for any period \( t = 0, \ldots, + \infty \). It will be assumed that start-of-period holdings of assets are given at predetermined levels. The central bank performs a transitory open market operation under the guise of a one-period increase of the growth rate of nominal balances, purchasing short-term public debt securities (or Treasury-Bills).

A temporary higher \( \mu_t \) implies higher present consumption for active agents, and a lower marginal utility of present consumption, leading to a higher marginal rate of inter-temporal substitution. With no changes in the discount rate, short-term nominal interest rates will drop (the so-called “liquidity effect”). Assuming a transitory monetary shock without any significant change in the expected inflation rate for \( t+1 \), the direct effect is observed on the nominal short term interest rate \( i_t \). Under such assumption, full inter-temporal arbitrage causes future short term rates \( t \) and spot long term rates to drop, i.e. a downward move of the yield curve. The monetary shock implies a real monetization with an increase of the average expected consumption of rich households, and some of those rich and inactive agents will have incentives to become active traders. In case of persistent monetary shocks, it is not unlikely to observe the “Fisher effect”: when expected inflation and real interest rates move in opposite direction.

US historic evidence suggests the existence of the so called “liquidity effect” after expansive open market operations: an initial drop of short-term nominal interest rates, might revert afterwards. The slope of the yield curve can change, either with long term rates below initial levels, but with relative lesser changes than those observed in short-term rates, or with long term rates higher than the initial ones. Since Friedman (1968), empirical literature shows that money injections in US tend to induce an initial short rate fall, followed by a smaller rise, coupled with a rise of long-term rates. Real interest rates depend only on active traders’ participation (the “segmentation effect”). Because of asset trade, money circulates among rich and poor people, leading to a positive impact on the consumption of poor people, unless agents’ expectations make it advisable to hoard money.

Under a liquidity trap the term structure of interest rates does not respond to open market operations because: a) the transmission mechanism from monetary shocks to inflation expectations, from these to the term structure of interest rates, and finally to aggregate demand and real output does not work, and b) asset market psychology is bearish, resulting an excess supply for commodities and financial assets. Furthermore, it implies a low level for aggregate demand, and a “bearish” asset markets.

5. Some perils getting close to low interest rates
5.a. A liquidity trap case?

Short term interest rates are zero with excess supply of goods, and excess demand for T-Bills and for money (because of downside pressure on the price level), with the long term bonds market is either in equilibrium or with excess supply.

This case is similar to a liquidity trap: positive real interest rates won’t change with open market activities on the short side of the yield curve. Money injection through payment of mature T-Bills and
market retirement of outstanding short-term debt won’t reduce their price. Therefore, in the short term agents expect a consumption fall because \( U_{t+1}^{ct} (c_{t+1}, \mu_{t+1}, \xi_{t+1}) < \beta U_{t}^{ct} (c_{t}, \mu_{t}, \xi_{t}) \). A reversion of such scenario requires higher expansionary real and/or monetary shocks in the immediate subsequent period. Under rational expectations it would increase the aggregate demand, pushing the “shadow” short-term rate to a positive level, but it will be also needed that such change assure that \( c_{t+1} > c_t \) so that the inverse of the gross interest rate be equal to the intertemporal marginal rate of substitution.

Observed short-term rate will be zero with commodities excess supply. If this scenario results from a reversible, transitory shock, aggregate demand is expected to recuperate itself, but at zero interest rates, and negative consumption expectations, it does not look so easy. The deflationary gap at zero interest rates is translated into high real interest rates in presence of expected price deflation.

Everybody (not only the government) wish to be short term debtor, but nobody wants to be a lender at zero interest rates. Agents want to be lenders for long term horizon, borrowing money at zero short-term rates. However, nobody has incentive to lend money at zero interest rates, unless there be a certain deflation rate for short term rates. Therefore, a low observed income at \( Y_{t}^{D} \) is consistent with excess supply for goods and excess demand for assets. Excess demand for short term debt implies a market price (interest rate) that is lower (higher) than equilibrium. Zero short term rate with positive long term rates induce agents to be a short term net borrower, with zero demand for long term loans, but it is not sure to be a “general” equilibrium.

Once interest rates are zero, expansionary open market operations change nothing. Long term interest rates might become lower with expansionary open market activities, but it would probably require increasing amounts of “helicopter money” for both stimulate aggregate demand and reverse depressive expectations.

The higher the long term interest rate, the higher the required money issuance (and probably the longer the required time span for monetary expansion) to cause a short-term interest rates hike. Once short-term rates be higher than zero, it is a matter of judgment and expertise to perform a combination of open market activities and fiscal measures to “normalize” the economy and expectations, avoiding a liquidity trap.

When nominal balance increases are met by the (speculative) demand for money, every long-term debt excess supply is matched by money excess demand, and in such case market activities with long term bonds are innocuous (not merely “neutral”).

If it is not the case under close-to-zero interest rates, it is legitimate to guess that the problem rests on a disequilibrium stance in the bond market, rather than on an “infinite” money demand, or an infinite interest-elasticity for money demand. In other words: it appears as the existent of an intertemporal coordination problem in bond market, rather than in money market. Almost surely, further information requires either a change in the model specification) appear as needed to address it. However some cases that put in jeopardy monetary effectiveness emerge clearly with this version of the AAK model.

5.b. Temporary output fall when price level drop is perceived as permanent

Let’s assume that trees start to provide less and less fruit in the islanders start to expect permanent lower dividends. It implies an expected output drop, with negative impact on both current and future private consumption. Equation (39) from Appendix C shows that the negative impact on current and future consumption leads to higher real interest rates when the consumption drop is expected to be permanent. Downward expected dividends drive down disposable income for all households.

Therefore, inflation rates, may turn in expected price deflation, with the simultaneous decrease of equilibrium long term interest rates (“Fisher effect”). Under deflationary expectations monetary shocks are ineffective. Capital gains become negative, therefore the really profitable alternative is to hoard nominal balances.

Trading volume and asset prices drop down, and the number of active households decreases too (“segmentation effect”). Asset market turnover decreases and those who liquidate their assets at a relatively low price will expect lower prices, reinforcing deflationary expectations. Therefore, expected
holding returns from previously purchased assets, and the longest-term yields-to-maturity of newly issued bonds must go down, flattening the yield curve:

**Nominal interest rate** ($i_t$)

- **a**: liquidity effect
- **b**: Fischer effect

![Figure 7.](image_url)

Shortest term yields will accompany the fall of longest term interest rates, and (nominal) total rates of return descend, pushing down short-term and long-term interest rates, but not necessarily to zero. In other words, this is a sort of liquidity trap at positive interest rates. Arbitrage relationships are useful here. Given the pricing kernels from Appendix (2):

$$R_{t+1}^{-1} = \beta \frac{U^T(c_{z,t}((\xi_{t,1}, \mu^{-1}, \mu^-)))}{U^T(c_{z,t}((\xi_{t,1}, \mu^-)))} = S_{t+1}$$

$$S_{t+1} = \sum_{k=1}^{n} S_{t,k}E_{t,k}S_{t+1,k} + \sum_{k=1}^{n} \prod_{j=1}^{k} \rho_j(S_{t,j}, S_{t+1,j}) \sqrt{\text{Var}S_{t,j}} \sqrt{\text{Var}S_{t+1,j}}$$

$$= \sum_{k=1}^{n} S_{t,k}E_{t,k}S_{t+1,k} + \sum_{k=1}^{n} \prod_{j=1}^{k} \text{cov}(S_{t,j}, S_{t+1,j}) = \sum_{k=1}^{n} S_{t,k}E_{t,k}S_{t+1,k} + \sum_{k=1}^{n} \prod_{j=1}^{k} E_j[(S_{t,j} - E_{t,j})(S_{t+1,j} - E_{t+1,j})]$$

In case of constant conditional successive-return covariances, the expectations hypothesis interprets such phenomenon as the result of downward expectations for future interest rates. When expected consumption and future asset prices drop with decreasing nominal short-term interest rates, real interest rates must go up. When active households perceive deflation as permanent or "very" persistent, both short and long-term interest rates go down, and agents gain a positive real yield from nominal money holdings. If deflation rate becomes higher than nominal interest rates, open market operations are unfeasible: asset trading stops and rich agents tend to become "inactive". According to Equation (20), it is intuitive to interpret it as a transition where variances of the whole range of interest rates go down. With arbitrated spot and forward markets, and zero covariances interest rates dynamics becomes:

$$R_{t+1}^{-1} = \sum_{i=1}^{n} R_{i,t}^{-1}E_{i,t+1}$$

$$R_{t+1}^{-1} = \sum_{i=1}^{n} (1 + i_{t+1})^{-1} = \sum_{i=1}^{n} (1 + i_{t+1})^{-1} E_{t+1}$$

From equation (21), an inverted yield curve implies decreasing future short term nominal interest, reflecting deflationary expectations for medium and long term horizons. Under full arbitrage, the eventual inversion of the yield curve requires some negative interest rates between some "intermediate" terms to maturity between the shortest rates and those of the longest maturities.

However such inversion involves “free lunches” in forward contracts for those periods when the implicit nominal forward rates of interest are negative. It looks like open market activities’ ineffectiveness is rooted in active agents’ perceptions, that prevents them to invest in asset market transactions, and it does not require necessarily zero interest rates. The Euler Equation for wealthy households’ optimisation implies:

$$U'(c_{z,t}(\xi_{t,1})) = \beta \frac{R(1+\pi_{t,1})U(c_{z,t+1}(\xi_{t+1}))}{(1+\pi_{t+1})}$$

From Equation (49):

$$U'(c_{z,t}(\xi_{t,1})) = \beta \frac{(1+i_{t})U'(c_{z,t+1}(\xi_{t+1}))}{(1+\pi_{t+1})}$$
When cash-in-advance and wealth constraints are binding, and taking into account active households for 1 period horizon, the first order condition, under rational expectations becomes:

\[
U'(c_{t+1}(\xi_t)) = \beta \frac{(1 + i_{t+1}) \cdot U'(c_{t+2}(\xi_{t+2}))}{(1 + \pi_{t+1})}
\]

(60)

When money demand is inflation-elastic, the signs are the opposite. Under elastic real balances demand, future inflation forecasts follow an "accumulative" process, as suggested by Howitt (1992).

Assuming that forecasting mechanics for a random variable \( x_t \) is:

\[
\begin{align*}
\pi_t &= \pi_t(x_{t+1}(\pi_{t+1}, \pi_{t+2})) \\
\frac{\partial \pi_t}{\partial \pi_{t+1}} &> 0
\end{align*}
\]

(67)

In case of an inflation-inelastic money demand, the signs are the opposite. Under elastic real balances demand, future inflation forecasts follow an “accumulative” process, as suggested by Howitt (1992). Assuming that forecasting mechanics for a random variable \( x_t \) is: \( \{x_t\}_{t=0}^{\infty} \) and that \( x_{t+1} = h(\pi_{t+1}) \) with \( h > 0 \).

When \( E_\pi x_{t+1} < x^* \), the sequence \( E_\pi (x_{t+1}) = J_t(\{x_t\}_{t=0}^{\infty}) \) is strictly decreasing, and when \( E_\pi x_{t+1} > x^* \), it is strictly increasing. If active agents underestimate positive monetary or real shocks the subsequent cumulative dynamics changes the optimal transition towards equilibrium. If agents expect permanent deflation and lower interest rates, from date \( t \) to \( t+1, t+2, \ldots, t+n \), a temporary tight monetary contraction might induce a short term increase in 1-period interest rates, but there are no reasons to expect changes in either current or future consumption.

The right side of Equation (62) is also the opportunity cost of consumption: its increase will induce a lower consumption at present date \( t \). That effect is unavoidable when agents expect a future monetary contraction, and because of that today’s active agents reduce the amount of liquid funds invested in asset trading for next period, given expected monetary contraction and price deflation paths. Therefore, the central bank might attempt to stop a deflationary spiral with expansionary monetary
shocks. However, the zero-lower bound for nominal interest rates is binding, whatever happens with the price index. It might imply that all interest rates (both short and long rates) become zero. However it is not the case

**Proposition I**

Under full arbitrage, non-satiation in consumption, and expected consumer price index deflation, active traders’ intertemporal equilibrium implies a positively sloped nominal yield curve.

**Proof:** See Appendix B.

**Additional remark:** other way to see it is as a case where negative shocks are not expected to change. Zero interest rate requires to assume a process such that $s_{t+2}$ and posteriors maintain $s^{th} = s^{t+1}$ (i.e. 2) and that satiation point was reached in $t$, but there is not a strictly equilibrium price vector that support any equilibrium there. Given (7), and the non-negativity of Lagrange multipliers, prices, discount factor and marginal utilities $(i_t(s_t)≥0)$, otherwise intertemporal contracts cannot exist. Therefore, neither spot nor forward nominal rates, can be lower than zero. As a consequence, the yield curve must be either rising or flat, never “inverted”.

**Corollary**

Under Proposition I assumptions, intertemporal equilibrium with permanent expected price deflation is unfeasible under zero interest rates.

**Proof**

Permanent price deflation requires $s_{t+i} > s_{t+i+1}$. Therefore, $D_t D_{t+1}$ for every $t$, and intertemporal equilibrium cannot be supported by zero interest rates. ■

**6. Concluding remarks**

The AAK setup engenders the “liquidity effect” thanks to endogenously generated market segmentation frictions, subject to a crucial assumption: agents allocate cash between goods and assets before observing the open market operation and its size. In such model the liquidity effect because of monetary persistent shocks are transitory. If households do hold cash every period, or spend their money “slowly” over time, it is highly likely to expect amplification effects from open market operations: the price level cannot respond equiproportionally to money supply changes. In such case aggregate real balances won’t be unchanged, and under constant aggregate output, a monetary expansion implies an increase of real balances, but the circulation velocity and nominal interest rates will fall.

In such model, under constant output it is optimal to spend all cash balances in every period, velocity is constant and money is neutral. However, such requirements do not appear as necessary in this modified version of the original setup.

It is apparent, from the conventional open market operations can become ineffective when lower liquidity for asset trading, imply lower consumption prospects for wealthy agents, and a further fall of aggregate demand. In this use of the modified AAK setup, it is shown that zero-interest rates cases are feasible, and that a close-to-liquidity trap solution emerges when agents perceive negative demand shocks as permanent. Furthermore, it is also shown that it is optimal to hold cash between periods when the expected inflation rate is negative.

Such result is a significant departure from the original A KK model, given that it provides sufficient conditions so that ensure households never carry over cash across time, in either goods or asset markets. In such case, no deviant agent is intended to carry over cash balances, and such result depends on a critical assumption: that their minimum utility level from asset trading is a fixed proportion of that one of a “representative” average agent $(c^A(μ') = \theta c^A(μ'))$, under constant risk aversion. Some slight changes in such assumptions were helpful enough to show some conditions so that households might carry over cash from one period to others. Specifically, this paper showed that persistence of low output-low interest rates can emerge directly because of the holding of unspent real balances given persistent expected deflation.

With regards to real evidence about “inverted” yield curves, it should reflect factors present in related the “preferred habitat” or “segmentation” hypotheses, but there are no convincing arguments or literature that turn easy to be found as compatible with this version of the AAK model.
Active open market activities interventions might be effective to recuperate aggregate demand when perceived as the starting path towards of a new equilibrium path with higher interest and inflation rates. It implies, a switch of pessimism among “rich” households (not necessarily among all households), in order to increase the liquidity invested by wealthy households in asset portfolios. However, the discussion of these issues might probably require that the AAK model be subject to deeper adaptations. A significant one, not addressed here, is related to the response of the number of active and inactive agents to state variables and interest rates. Nevertheless, such a matter has been left subject for further future research.
APPENDIX A

Fiscal shocks \( \{ g_t \} \)
Initial monetary shocks\( \{ \mu_t \} \)
Real shocks
\[ \xi_t \]

Initial endowments
Shopper end of day assets
\[ S_0 \]
\[ M_0^r = M_{t,obs}^r - \Delta M_{t,obs}^r + \Delta M_{t,earned,obs}^r = M_t^r = p_t y_t = \begin{cases} p_t \cdot (c_t + x(\mu_t) + \gamma) & \text{if active} \\ p_t \cdot c_t & \text{if inactive} \end{cases} \]

Worker \( y_t \) Sells Pays Earns dividends dividends on the others’ trees
\[ p_t \cdot y_t(x_t) = \Delta M_t^r \]

Open market operations
\[ n_t \]

\[ B_0^p(n_0 + \Delta n_1) + s_0(q_0 + \Delta q_t) + \left[ n_t \Delta B_0^p + q_t \Delta s_t \right] \]

\[ B_1 \]

\[ M_1^p \]

\[ S_1 \]

\[ y_t \]

Figure 1. Wealthy households’ trading sequence

A.1. Endogenous market segmentation and exogenous shocks

Assume that money injections satisfy \( \hat{\mu}_{t+1} = \rho \hat{\mu}_t + \varepsilon_{t+1} \), where the hat indicates the time derivative of the logarithm of the variable (i.e. the relative variation rate). Coefficient \( \rho \) is the persistence of the money shock and \( \varepsilon_{t+1} \) is a normal, i.i.d. innovation with mean zero and variance \( \sigma^2 \). The expected inflation effect is given by the direct short term impact on a fixed endowment economy where money is not fully hoarded, but unlike the basic AAK model, real and nominal interest rates do not necessarily show the same persistence as money shocks. Let’s remember equation (31) from the wealthy households’ optimization:

\[ \beta \frac{U_{t+1}(c_{t+1}, \xi_{t+1}, \mu_{t+1}, \mu_{t+2})}{U_t(c_t, \xi_t, \mu_t, \mu_{t+1})} = R_t \]

From here:

\[ \hat{\mu}_t \Rightarrow \hat{\mu}_{t+1} \Rightarrow [\hat{\mu}_{t+2}]^{(a)} \]

\[ \downarrow \hat{\mu}_{t+2} \Rightarrow [\hat{\mu}_{t+2}]^{(b)} \]

Cases (a) and (b) are both feasible, even under the assumption of mean reversion. What we know is that the lower the \( \rho \) coefficient, the lower the persistence of the process (lower memory). Case (a) is the same than the AAK model: higher monetary shocks in \( t \), higher expected shocks for \( t+1 \) period. Therefore, a present monetary shock that increases present consumption may accelerate consumption growth rate between \( t \) and or \( t+1 \), lowering interest rates in \( t \). It implies a negative relationship between monetary shock (and inflation change) and real interest rates. Case (b) is not discarded, but the AAK result is held only when \( |\hat{\mu}_{t+2}| < |\hat{\mu}_{t+1}| \).

The ineffectiveness of open market activities cannot be ruled out, and a liquidity trap may emerge naturally in the same way that this kind of model exhibits well-behaved and realistic impact effects of open market activities. Cash-in-advance constraint is binding for “rich” households, either they are “active” or not in asset markets, but unlike the AAK model, the trading costs have not an effect
constrained to current consumption. Therefore, subsequent periods will be affected by current portfolio decisions. Households are identical, but Markovian shocks are idiosyncratic. The AAK model properly states that it will depend on the measured gain from switching from being inactive (“I”) to active (“A”). In term of utility, AAK original model states such gain as:

$$g(m + x, m) = \left[U(c_i) - U(c_j)\right] - U'(c_j)(c_j - c_i)$$  \hspace{1cm} (1)$$

Direct utility gain - Utility cost because of funds rotation in asset transactions

where $c_A$ is a fixed level of per capita consumption, corresponding to an arbitrary level of $\mu_k$.

Here it will be assumed that monetary and price level shocks affect indirectly to $z$ through their impact on the level of real balances:

$$z\left\{ \begin{array}{ll}
0 & \text{when inactive} \\
1 & \text{when active} 
\end{array} \right. \text{ where } c_{z,t}(\xi, \mu', \mu_{t+1}) = m_{x,t}(\mu', \pi', \pi_{t+1})$$. 

Trading is profitable when consumer’s utility is higher than the case when the agent is inactive, and

$$g(\bullet) = 0 \quad \text{when} \quad \begin{array}{l}
U(m_{x,t} + x - \gamma) - U(m_{x,t}) = U'(c_A) \cdot (c_A + \gamma - m_{x,t}) \\
U(m_{x,t} + x) - U(m_{x,t}) = U'(c_A) \cdot (c_A + \gamma - m_{x,t})
\end{array}$$

There is a critical “high” value for $m_{z,t}$ so that asset trading is performed when $g(\bullet) > 0$. On the other side, there is a critical “low” value for $m_{x,t}$, so that asset trading is only feasible for income, real balances and consumption below such threshold. The function shows a “U” shape, with two critical values where $g(\bullet) = 0$. Given a level of active consumption there is an interval where the shopper is not active in asset trading, but such behaviour will depend on the determinants of those critical values and of the position of this function: money shocks and dividend shocks. It is observed for an infinitesimal increase of $m$, i.e., when $\gamma = 0$, all agents become active, and the minimum of the $g$ function is observed when $m = c_A$. Households trade assets when $g(\bullet) > 0$, i.e.:

$$g(\bullet) = \left[U(c_J) - U(c_i)\right] - U'(c_J)(c_J - c_i) = \\
[U(m_{x,t} + x) - U(m_{x,t})] - U'(m_{x,t} + x)[m_{x,t} + x - m_{x,t}] > 0$$

or:

$$g(\bullet) \left\{ \begin{array}{ll}
> 0 & \Leftrightarrow U(m_{x,t}(\mu', \pi', \pi_{t+1}) + x(\mu_i)) - U(m_{x,t}(\mu', \pi', \pi_{t+1})) > U'(m_{x,t}(\mu', \pi', \pi_{t+1}) + x(\mu_i)) \\
< 0 & \Leftrightarrow U(m_{x,t}(\mu', \pi', \pi_{t+1}) + x(\mu_i)) - U(m_{x,t}(\mu', \pi', \pi_{t+1})) < U'(m_{x,t}(\mu', \pi', \pi_{t+1}) + x(\mu_i))
\end{array} \right.$$

In other words, $g(\bullet)$ is positive (negative) when the average utility of becoming active, measured on the average consumption between inactivity and a target consumption assuming active trading, is higher (lower) than the marginal utility computed in that point. In absolute terms, it corresponds to a consumption lever that is in absolute value higher than $c_A \pm \gamma$, when transaction costs are not zero. The critical points where the $g$ function is zero are those where the infinitesimal moves from activity to inactivity are such that average and marginal utilities are the same.
The effect of monetary shocks on the $g(\cdot)$ function depends on the impact monetary shocks on real balances at any date $t$. Under low inflation (i.e., real balances demand is price-inflation inelastic):

$$\frac{\partial g}{\partial \mu} = \left( \frac{\partial U_{c_A}}{\partial \mu} - \frac{\partial U_m}{\partial \mu} \right) - \frac{\partial U_{c_A}}{\partial \mu} \left( c_A - m_{T,t}^p \right) + U_{c_A} \left( \frac{\partial c_A}{\partial \mu} - \frac{\partial m}{\partial \mu} \right) =$$

$$= \left( \frac{\partial U_{c_A}}{\partial \mu} + \frac{dx}{d \mu} + \frac{\partial U_{c_A}}{\partial \mu} \right) - \frac{\partial U_m}{\partial \mu} \left( \frac{dx}{d \mu} + \frac{\partial U_m}{\partial \mu} \right) \left( c_A - m_{T,t}^p \right) + U_{c_A} \left( \frac{\partial m_{T,t}^p}{\partial \mu} + \frac{dx}{d \mu} \right)$$

$$= \frac{dx}{d \mu} + \frac{\partial U_m}{\partial \mu} \left( c_A - m_{T,t}^p \right) + U_{c_A} \left( \frac{\partial m_{T,t}^p}{\partial \mu} + \frac{dx}{d \mu} \right)$$

For $c_A > m_{bl}$, higher $\mu$ leads to a higher $g$. When $c_A = m_{bl}$, $\frac{\partial g}{\partial \mu} = \frac{dx}{d \mu} + U_{c_A} \left( \frac{\partial m_{T,t}^p}{\partial \mu} + \frac{dx}{d \mu} \right) > 0$.

When $c_A < m_{bl}$, under the assumption of non-satiation for active traders the sign is also positive, but of lower magnitude than in the previous case.

Therefore $\text{sgn} \left( \frac{\partial g(\cdot)}{\partial m_{T,t}} \right) = \text{sgn} \left( \frac{\partial g(\cdot)}{\partial \mu} \right)$ and the impact of monetary shocks is that of the original AAK model:

**Figure I:** Inactivity zone for "rich" households

**Figure II:** Positive monetary shocks incentive asset trading, reducing the distance between thresholds $m$ and $m'$. 
Under a high inflation regime, money demand is price inflation-elastic, therefore:

$$\frac{\partial g}{\partial \mu} = \left( \frac{\partial U_{c,A} \partial m^p_t \partial x}{\partial m^p_t} + \frac{dx}{\partial \mu} - \frac{\partial U_{c,A} \partial m^p_t}{\partial \mu} \right) - \frac{\partial U_{m} m^p_t}{\partial \mu} \left( \frac{dm^p_t}{\partial m^p_t} + \frac{dx}{\partial \mu} \right) \left( c_A - m^p_t \right) + U_{c,A} \left( \frac{dm^p_t}{\partial m^p_t} + \frac{dx}{\partial \mu} \right) \left( c_A - m^p_t \right)$$

$$\frac{dx}{\partial \mu} \left( \frac{dm^p_t}{\partial m^p_t} + \frac{dx}{\partial \mu} \right) \left( c_A - m^p_t \right) + U_{c,A} \left( \frac{dm^p_t}{\partial m^p_t} + \frac{dx}{\partial \mu} \right) \left( c_A - m^p_t \right) =$$

$$\left( \frac{dm^p_t}{\partial m^p_t} + \frac{dx}{\partial \mu} \right) \left( c_A - m^p_t \right) + U_{c,A} \left( \frac{dm^p_t}{\partial m^p_t} + \frac{dx}{\partial \mu} \right) \left( c_A - m^p_t \right)$$

(3)

When \( c_A > m_{2,t} \), the second factor is negative (the higher the difference \( c_A > m_{2,t} \), the greater the effect on \( g \)) monetary shocks cause \( g \) to rise. On the other side, when \( c_A < m_{2,t} \), the effect is positive, but of lower magnitude than in the previous case. It is because higher nominal balances are partly demanded for commodity purchases (flight from money, and higher velocity of circulation) and partly for asset trading (when stock and bonds provide a hedge against inflation). The \( g(f) \) function moves like in figure II. The other side of the coin is the inverse relationship between nominal yields and real interest rates, from Lucas (1990):

$$\frac{d\ln z_{it}}{dt} \approx -\frac{d\ln x_{it}}{dt} = \frac{d\ln z_{it}^*}{dt}$$

Monetary expansionary shocks induce an upward move of the \( g \) function, inactive agents become active, and there is an increase of liquidity poured on asset markets. Therefore, nominal interest rates go down (the “liquidity effect”). In the liquidity trap agents perceive that asset purchasing is less profitable than hoarding money. Monetary shocks under the guise of open market activities are ineffective because there is not demand for government debt, and interest rates do not change. If they are zero, stay at zero. But if they are higher than zero, they will not change. Like in a “normal” scenario, the interest rate level is endogenous to expectations on monetary (among others) shocks.

A.2.. Stocks, flows and money demand

It will be assumed that the government provides lump sum transfers according to the perception of the need of such transfers by poor people. Budget constraints for the government, when public expenditure is \( g_t \), and taxes are called \( \tau_t \):

$$g_t < y_t(\xi_t)$$

$$g_t = \tau_t + \frac{b_{1,t}}{R_t} + \frac{M_{1,t} - M_{1,t}}{P_t} - b_t$$

$$\tau_t = \left[ g_t - b_1 \int n_{1}(\mu_{1,t}, \mu_t) d\mu_{1,t} - \frac{b_{1,t}}{P_t} - \frac{M_{1,t} - M_{1,t}}{P_t} \right]$$

$$g_0 = \tau_0 + \frac{b_1}{R_0} + \frac{M_{0} - M_{0}}{P_0}$$

$$b_0 = 0$$

From here, active households’ constraint is:

$$\frac{M_{1,t} \gamma}{P_t} + \tau_t + s_{1,t} \int q_{1}(\xi_{1,t}, \xi_t) d\xi_{1,t} + b_1 \int n_{1}(\mu_{1,t}, \mu_t) d\mu_{1,t} =$$

$$= \frac{y_{1,t}(\xi_{1,t})}{P_t} P_{1,t} + s_{1,t} \int q_{1}(\xi_{1,t}, \xi_t) d\xi_{1,t} + \frac{b_{1,t}}{P_t} + \frac{M_{1,t} - P_{1,t}(c_{1,t} - \tau_{1,t}) - \gamma}{P_t}$$

(superscript \( p \) indicates nominal balances held by the private sector)
Assuming \( s_i = s_{i+1} = 1 \), and inserting the cash-in-advance constraint:

\[
(c_i + \frac{\gamma}{P_t}) + g_i = \frac{M_{t+1}^P - M_t^P}{P_t} + y_{i-1} \frac{P_{i-1}}{P_t}
\]

Active agents buy stock shares and government bonds:

\[
M_{t+1} = M_t = P_t \left( m_t^F + m_t^B \right) = P_t \left( m_t^F + m_{t+1}^F + m_{t+1}^B \right) = P_t (c_i + g_i - \tau_i) + P_t \gamma \quad \text{and} \quad P_t m_t^B = P_t g_i,
\]

\[
m_t^F + b_j \int n_j (\mu_{i-1}, \mu_i) d\mu \left[ -b_{t+1} n_j (\mu_{i-1}, \mu_i) d\mu, \right] = c_{i-1} - \tau_i - s_i \int q_j (\xi_{i-1}, \xi_i) d\xi_i + \gamma
\]

\[
m_t^F + b_j \int n_j (\mu_{i-1}, \mu_i) d\mu \left[ -b_{t+1} n_j (\mu_{i-1}, \mu_i) d\mu, \right] = c_{i-1} + \gamma + \gamma
\]

\[
\text{where} \quad m^F_t = c_{t-1} - \tau_i - [x_i - \gamma]
\]

In the case of inactive rich agents and poor agents:

Rich inactive

\[
m^F_t = c_{t-1} - \tau_i
\]

Poor

\[
m^B_t = c_{i-1} + \xi \tau_i
\]

In such case, aggregating across agents, assuming \( s_i = s_{i+1} = 1 \), and inserting the cash-in-advance constraint we get the basic identities for the private sector:

Rich active

\[
\sum_{i=1}^{k} m^F_{t+1} = \sum_{i=1}^{k} \left[ c_{i+1} - \tau_{i+1} - \left( x_{i+1} - \gamma_{i+1} \right) \right]
\]

Rich inactive

\[
\sum_{i=1}^{k} m^F_{t+1} = \sum_{i=1}^{k} \left[ c_{i+1} - \tau_{i+1} \right]
\]

Poor

\[
\sum_{i=1}^{k} m^B_{t+1} = \sum_{i=1}^{k} \left[ c_{i+1} + \xi \tau_{i+1} \right]
\]

Aggregating the three sets of households and the government financial constraint, and netting purchases and sales of stock shares:

\[
\sum_{i=1}^{k} m^F_t + \sum_{i=1}^{k} m^F_t + \sum_{i=1}^{k} m^B_t = \sum_{i=1}^{k} \left[ c_{i+1} - \tau_{i+1} - \left( x_{i+1} - \gamma_{i+1} \right) \right] + \sum_{i=1}^{k} \left[ c_{i+1} - \tau_{i+1} \right] + \sum_{i=1}^{k} \left[ c_{i+1} + \xi \tau_{i+1} \right]
\]

\[
m_t^F = \gamma_i = \sum_{i=1}^{k} \left[ c_{i+1} \right] + \sum_{i=1}^{k} \left[ \tau_{i+1} \right] + \sum_{i=1}^{k} \left[ x_{i+1} - s_{i+1} \right]
\]

From here:

\[
m_t^F + m_t^B = c_i + g_i + \sum_{i=1}^{k} \left[ c_{i+1} + \gamma_{i+1} \right] \quad \Rightarrow \quad \frac{M_t}{P_t} = 
\]

Assuming a particular point without monetary shocks \( (M_t = M_0) \) but real balances and real output do not change:

\[
\frac{M_{t+1}(\mu_{t+1})}{P_{t+1}} = Y_t \frac{P_t}{P_{t+1}} + \frac{K \gamma}{P_t} \quad \Rightarrow \quad \frac{P_t}{P_{t+1}} = \frac{M_{t+1}(\mu_{t+1})}{P_{t+1}} - \frac{K \gamma}{Y_t(\xi_t)} \quad (i)
\]

\[
m_t^B = \frac{M_{t+1}}{P_{t+1}} = \frac{Y_t(\xi_t)}{1 + \pi_{t+1}} \frac{P_t}{P_{t+1}} + \frac{K \gamma}{P_t(1 + \pi_{t+1})} \quad (ii)
\]

Provided that poor households cannot purchase any hedge against expected inflation, higher prices reduce consumption of both poor and rich "inactive" households. Therefore, with a price level subject to monetary shocks that follow a Markov process, but, given that \( M_{t+1} = M_t \) without monetary nor real shocks, on equation (52):

\[
m_t^B = \frac{M_t}{P_t} = \frac{Y_t(\xi_t)}{1 + \pi_{t+1}} \frac{P_t}{P_{t+1}} + \frac{K \gamma}{P_t(1 + \pi_{t+1})} = m_t^B
\]
APPENDIX B
Proof of Proposition I
Under the assumptions of the basic model of Appendix B, each “active” household maximizes its utility
\[ E_0 \sum_{t=0}^{\infty} \beta^t \left[ U \left( c_{t, t} (s', s_{t+1}) \right) f(s') ds' \right] g(s_{t+1}) ds_{t+1} \]  
\[ s_i = (\xi_i, \mu_i), s' (\xi_i', \mu_i') \]  
subject to these constraints:

Non monetary wealth
\[ \theta_0 (s_i) = (x_0 (s_0) \cdot z(s_0)) + \sum_{t=1}^{\infty} D_i (s_i) \left( x_i (s_i) \cdot z(s_i) + \tau(s_i) - \alpha (P_i (s_{t-1}) \cdot y_{t-1} (s_{t-1})) \right) ds_i \]  
\[ z = \begin{cases} 0, & \text{if asset trading} = 0 \\ 1, & \text{if asset trading} \neq 0 \end{cases} \]  
(Assuming perfect substitutability between stock shares and 1-period discount bonds, D_i is the price at of a claim to one dollar to be delivered in t, in state s_t, and \( \alpha \) is the fraction of the household’s earned money used to pay for the broker’s service or transaction cost. The letter x refers to the liquidity used to pay for securities).

Nominal balances at the end of t=0:
\[ M_{Z,0}^r (s_o) \geq P_0 (s_0) \left[ c_{Z_0} (s_0, s_0) + (x_0 (s_0) + \gamma) \cdot z(s_0) - \tau_0 (s_0) \right] = \frac{1}{1 - \alpha} \cdot \left[ M_{Z,0}^r (s_o) + (x(s_0)) \cdot z(s_0) \right] \]  
\[ M_{Z,0}^r (s_o) : \text{initial holdings at period 0} \]  
Nominal balances at later dates:
\[ M_{Z,t+1}^r (s_{t+1}) = x_{t+1} (s_{t+1}) \cdot z(s_{t+1}) + M_{Z,t}^r (s_t) + (1 - \alpha) y_t (s_t) - P_t (s_t) \cdot c_{Z,t} (s', s_{t+1}) \]  
Cash-in-advance constraint:
\[ M_{Z,t}^r (s_t) \geq P_t (s_t) \cdot c_{Z,t} (s', s_{t+1}) \]  
The corresponding Lagrangean is:
\[ L = E_0 \sum_{t=0}^{\infty} \beta^t \left[ U \left( c_{Z,t} (s', s_{t+1}) \right) f(s') ds' \right] g(s_{t+1}) ds_{t+1} + \]  
\[ + \lambda_{x_0} (s_i) \left[ \theta_0 (s_i) - (x_0 (s_0) \cdot z(s_0)) - \sum_{t=1}^{\infty} D_i (s_i) \left( x_i (s_i) \cdot z(s_i) + \tau(s_i) - \alpha (P_i (s_{t-1}) \cdot y_{t-1} (s_{t-1})) \right) ds_i \right] + \]  
\[ + \lambda_{M_{Z,t}} (s_t) \left[ M_{Z,t}^r (s_t) - \frac{1}{1 - \alpha} \cdot \left[ M_{Z,t}^r (s_t) - (x_t (s_t)) \cdot z(s_t) \right] \right] + \]  
\[ + \lambda_{c_{Z,t}} (s_{t+1}) \left[ M_{Z,t+1}^r (s_{t+1}) - x_{t+1} (s_{t+1}) \cdot z(s_{t+1}) - M_{Z,t}^r (s_t) - (1 - \alpha) y_t (s_t) + P_t (s_t) \cdot c_{Z,t} (s', s_{t+1}) \right] + \]  
\[ + \lambda_{4_4} (s_{t+1}) \left[ M_{Z,t+1}^r (s_{t+1}) - P_t (s_t) \cdot c_{Z,t} (s', s_{t+1}) \right] \]  
Cash-in-advance constraint is binding, and money injections are through open market operations, so changes in the outstanding debt are equivalent to that of money stock.

The first order conditions for active agents (under the conventional assumption of monotonicity in the utility function):
\[ x_t : \quad - \lambda_{1_1} (s_t) D (s_t) + \lambda_{2_2} (s_t) \frac{1}{1 - \alpha} = 0 \]  
\[ M_t : \quad \lambda_{2_2} (s_t) - \int \lambda_{3_3} (s_{t+1}) ds_{t+1} + \lambda_{4_4} (s_t) = 0 \]  
\[ c_t : \quad \beta^t U' \left( c_{Z,t} (s', s_{t+1}) \right) + P_t (s_t) \left( \int \lambda_{3_3} (s_{t+1}) ds_{t+1} - \lambda_{4_4} (s_t) \right) = 0 \]  
From these first order conditions:
\[ D(s_t) = \left( \frac{1}{(1 - \alpha) \lambda_{1_1} (s_t)} \right) \left( \frac{\beta^t U' \left( c_{Z,t} (s', s_{t+1}) \right)}{P_t (s_t)} \right) \]
Equation (6) yields the price in terms of dollars at \( t=0 \) of a claim of a dollar delivered in period \( t \). In case of investing a dollar (its cost was \( D(st) \)) from date \( t \) to date \( t+1 \), such dollar would be worth \( \int D_{t+1}(s)ds \), and the gross interest rate between \( t \) and \( t+1 \) is:

\[
1 + i(s_t, s_{t+1}) = \frac{D_t(s_t)}{\int D_{t+1}(s)ds_{t+1}}
\]

From here:

\[
i(s_t, s_{t+1}) = \frac{D_t(s_t)}{\int D_{t+1}(s)ds_{t+1}} - 1
\]

Expected price deflation means \( s_{t+1} > s_t \) therefore \( D_t(s_t) \neq \int D_{t+1}(s)ds_{t+1} \). From here, the one-period nominal interest rate must be non-zero for any \( t \), leading to a permanent fall of asset market prices. Because of simple arbitrage, non-negative forward nominal rates turn the yield curve "inversion" unfeasible. ■
References


Footnotes


2 The Federal Reserve Act (section 2A) states: “The Board of Governors of the Federal Reserve System and the Federal Open Market Committee shall maintain long run growth of the monetary and credit aggregates commensurate with the economy’s long run potential to increase production, so as to promote effectively the goals of maximum employment, stable prices, and moderate long-term interest rates”. The third of these goals has been forgotten for long, but it also results from the vagueness of the adjective “moderate” long-term interest rate, opening the way to multiple interpretations.


4 The $\iota$ term is a stochastic i.i.d. uncorrelated error at forecasting future short-term interest rates with zero mean and constant variance.

5 Benhabib, Schmitt, Groh and Uribe (2002) study in greater detail the interrelation between monetary and fiscal policy to avoid and to exit from a liquidity trap at zero nominal interest rates.

6 Leijonhufvud (2003) remarked that it requires an underlying effective demand failure. The Japan case illustrates that it might emerge because of the huge amount of non-performing loans after a financial crash. In that case the acute problem to solve is the reconstruction of mutual trust. However, he remarked that “the Japanese problem is neither household income-constraints nor a ‘liquidity trap in the usual sense’.

7 Some overviews are those by Humphrey (2003), Bordo, Lane and Filardo (2003), Bordo and Filardo (2004), and Borio and Filardo (2004). Goodhart and Hoffman (2004) emphasized that both the Great Depression and recent Japanese deflation were preceded by credit contraction and a significant fall of real estate prices. Borio and Redish (2003) distinguished supply-side led and demand-side led deflations, but they revealed that 19th century data reveals a combination of supply and demand shocks in deflation episodes.

8 Several estimates show that US population has not been typically “active” in asset markets. Using data from the 1989 Survey of Consumer Finances, Mulligan and Sala-i-Martin (1996) unveiled that 59% of US households did not hold any interest-bearing assets, and it was also the case for 53% of those who held cheque accounts. With the Panel Study of Income Dynamics from the University of Michigan Vissing-Jørgensen (2002) has showed that between 1968 and 1993 only 21.66% of US population holds stocks, and 31.34% were bondholders, and for the United Kingdom case, Attanasio, Banks and Tanner (2002) have also found significant evidence of limited participation in asset markets.


10 The history of dividends up to t is $Y_t = \{y_{t+1}, \ldots, y_T\}$, and $\xi$ is the history of stochastic idiosyncratic shocks up to date t.

11 “Fruits” are assumed as perishable, and with no migration between islands (except to the centralized market), no “palm-tree” might be seized in case of loan defaults.

12 Real-life evidence persistently show that risk and returns of stocks tend to be higher than those of T-Bonds, and these, in time, exhibit higher return and risk indicators than T-Bills.

13 Condition c) implies $R^{-1}_\beta = \int_{1}^{T-j} q(f) dtx_{t+j}$. When $y_t$ is indexed on the entire history of $x$ up to $t$ ($x_t$), this procedure is not admissible, because interest payments at $t$ depend upon the gross composite interest rates applied during the whole history up to $t$.

14 Following Sargent (1987), let be $y_t^j$ the current price of j-step-ahead contingent claim price of a unit of the consumption good in period $H_j$ (i.e., contingent on the occurrence of the state $x_j$ at period $t+j$). When $y_t = f(x_t)$, in that case it is valid to resort to this simplification for public debt: $\int_0^{T-j} \int y(x_t) n(x_{t-1}, x_{t-2}) \ldots n(x_1, x_0) dx_{t-1} \ldots dx_1 = \int y(x_t) n(x_{t-j}) (x_t, x_0) dx_t$.

15 This constraint implicitly assumes that households’ change in monetary balances are perceived under the form of a grant from the government (a lump sum transfer $g_t$ per “poor” household), so in the aggregate economy, Walras’ law is consistent with the observance of a binding transversality constraint. The access to grants is essentially history dependent for the time path of $\zeta (\zeta)$, but it is irrelevant to compute the relevant pricing kernels of this problem.

16 In a broad sense, it summarizes all trading costs (taxes inclusive) as fixed. Jouini, Kallal and Napp (2001) explore no-arbitrage cases in presence of fixed trading costs. Alvarez, Alkens and Kehoe (2006) have extended the primary AAK model to an open economy environment including state-dependent transaction costs.

17 It is assumed full arbitrage between stock and bond markets. Some evidence (see Jovanovic and Rousseau (2001)) suggests that it might be unfeasible because of market segmentation, so that co-movements between bond and stock prices could be nonexistent in real life markets.

18 Kazemi (1993) has shown, that in a fully Markovian economy, longest term interest rates depend on present t-date monetary shock. Empirical evidence, however, does not look to support such association. Recent revision by Duffee (2006) of historical series for two long periods (1960-2003, without the “Volker era” from mid 1979 through 1983) shows a remarkable absence of link between inflation (generally associated to monetary shocks), and mid-long term rates.

19 Real life shows that particular investor classes could be associated to specific risk-return profiles and investment “philosophies” that justify the segmented markets theory. Duffee (1996) found evidence that reflects it at comparing money market funds and other investors’ investments.

20 See Manuelli and Sargent(1987)). In Lucas (1978) the value function must be unique, bounded, continuous, nonnegative, and an increasing, concave function of shareholdings. Transaction costs may engender singularity properties in threshold points between activity and no activity in asset trading. To avoid that problem it can be assumed that the second derivative of both utility and value functions are continuous at those points. This is an additional condition known as smooth pasting or “super contact” condition (see Dumas (1981)).

21 Duffie and Zame (1989) have shown that in the CCAPM model, the short term (optimal) interest rate is the real equilibrium short term interest rate but not necessarily the government’s target.

22 Evidence of this effect was shown by Barr and Campbell (1997) with historical series from the United Kingdom.
AAK have shown that segmentation effect dominates the “Fisher effect” when the lower is the persistence of money growth. A transitory shock leads to a larger drop in the real interest rate than in case of a persistent shock on money growth. For sake of simplicity, the model does not consider sources of discontinuity or difficulties to project a continuous yield curve, capturing all terms to maturity with no exception from overnight transactions to perpetuities. An “inverted” yield curve does also signal downward future stock share prices. See equation (47). See Lucas (1990). Milani (2005) provides an alternative model to show the endogenous nature of the disturbing persistence of the bearish psychology that prevails in the transition from “normal times” to a liquidity trap, with inertia of learning mechanics.