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A GENERALIZED VARIABLE ELASTICITY OF SUBSTITUTION PRODUCTION FUNCTION WITH AN APPLICATION TO THE NEOCLASSICAL GROWTH MODEL

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A Generalized Variable Elasticity of Substitution Production Function with an Application to the Neoclassical Growth Model

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Abstract

This paper develops a generalization of a class of variable elasticity of substitution production functions introduced by Revankar (1971). The generalized function includes as special cases the more representative cases of the CES family that were absent from the original formulation. A complete characterization of the new class is provided, together with an application to the neoclassical growth model.

Resumen

El presente trabajo desarrolla una generalización de la clase de funciones de producción con elasticidad de sustitución variable introducida por Revankar (1971). La función generalizada incluye como casos especiales a los casos más representativos de la familia CES que no formaban parte de la formulación original. Se incluyen una caracterización completa de la nueva clase y una aplicación al modelo neoclásico de crecimiento.

Keywords: Variable elasticity of substitution; Production function; Solow-Swan model.

JEL classification: D24, E13, O40.

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1 Introduction

Revankar (1971) introduced a class of variable elasticity of substitution (VES) production functions that is generalized in the current paper. The following eloquent passages are a magnificent description of what motivated this article and the objectives pursued.

> "Once we recognize that σ may be a variable, we at once face a variety of ways in which to choose the functional dependence of σ on output and/or inputs. Preferably, of course, this choice should be such that it permits testing of a constant σ, so that the production function corresponding to the choice made is a generalization of the CES, *in very respect*. But a main constraint for such a choice may be the desire for a convenient economic interpretation of the selected behavior of σ and for a resulting production function that is empirically manageable and economically insightful."
>
> "Of course, it remains to be seen if it exists a suitable choice for the functional form of σ that permits a convenient CES generalization."

The article is organized as follows: Section 2 gives a brief description of the VES class introduced by Revankar; Section 3 provides a complete characterization of the generalized VES (GVES) class developed in this paper; Section 4 applies the GVES production function to the Solow-Swan model; and Section 5 concludes.

2 Revankar's VES production function

Consider the following VES specification $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ introduced by Revankar (1971), assuming constant returns to scale,

$$F(K,L) := \begin{cases} AK^{\alpha\nu}\left[L + (\nu-1)\,K\right]^{1-\alpha\nu} & 0 < \nu < \alpha^{-1},\, \nu \neq 1 \quad \text{(1a)} \\ AK^{\alpha}\,L^{1-\alpha} & \nu = 1, \quad \text{(1b)} \end{cases}$$

where $0 < \alpha < 1$ and $A > 0$. This specification has the Cobb-Douglas as canonical form which is obtained by setting $\nu = 1$ in (1a).

A distinguishing feature of Revankar's formulation is an elasticity of technical substitution that varies linearly with capital per worker

$$\sigma_V = 1 + \left(\frac{\nu-1}{\alpha\nu}\right)\frac{K}{L}, \qquad K > 0,\ L > 0. \tag{2}$$

Therefore, this production function has a unitary elasticity of substitution at the origin, i.e., $\sigma(0) = 1$, that is strictly increasing on $[0,\infty)$ if $\nu > 1$, and it remains constant if $\nu = 1$. In the case that $0 < \nu < 1$, the elasticity is strictly decreasing on a certain range of the capital-labor ratio, say $[0,\bar{k})$, and vanishes at $\bar{k}$.

As can be seen from (2), capital per worker K/L (hence, output per worker Y/L) must be restricted to ensure that σ_V takes non-negative values. Then, for all $K > 0$ and $L > 0$ the following must hold:

$$\frac{K}{L} \leq \bar{k}_1 := \frac{\alpha\nu}{1-\nu}. \tag{3}$$

In particular, this condition means that for $0 < \nu < 1$ there is an effective *upper bound* on capital per worker, hence K/L must lie in the interval $[0, \bar{k}_1]$.[1] However, if $\nu = 1$ the upper bound is not binding since $\bar{k}_1$ in (3) tends to infinity, and the condition results in a *negative* lower bound for K/L if $1 < \nu < 1/\alpha$. Adopting the common assumption of non-negative values for capital, the relevant domain for K/L in these two cases is $[0, +\infty)$.

The properties of this *canonical* VES production function will be studied in detail as part of the generalized form introduced in this paper.

3 A generalized VES production function

In this section a generalized class of VES production functions (GVES) will be developed, based on Revankar's VES specification. The generalization allows for a non-linear relation between the elasticity of substitution and the capital-labor ratio. However, this relationship is still monotonic.

Consider the following functional form $\varphi_p : \mathbb{R}^2_{++} \to \mathbb{R}_+$

$$\varphi(K,L)_p := \begin{cases} K^{\alpha\nu}\left[L^p + (\nu-1)K^p\right]^{\frac{1-\alpha\nu}{p}} & -\infty < p < 1, p \neq 0, 0 < \nu < \frac{1}{\alpha} \quad \text{(4a)} \\ K^{\beta}L^{1-\beta} & p = 0, (1+\alpha)^{-1} < \nu < \alpha^{-1}, \quad \text{(4b)} \end{cases}$$

where $0 < \alpha < 1$ and $\beta := (\nu - 1 + \alpha\nu)/\nu$. The restrictions imposed on ν in (4b) guarantee that $0 < \beta < 1$, so this is a variant of the Cobb-Douglas form. For reasons that will be clarified later, further restrictions are needed before giving a precise definition of a production function based on (4a). Note that as $p \to 1$, this equation reduces to the canonical VES form presented in (1).

This simple way to generalize Revankar's VES function results from the combination of geometric, arithmetic and harmonic means, known as a *mixed mean*. We can think of (4a) as a weighted arithmetic or harmonic mean (depending on the sign of p) embedded into a weighted geometric mean, which reduces to the geometric mean in (4b) as p tends to zero. Capturing this feature of mixed means, production functions that correspond to these functional forms are usually referred as "nested production functions" in the economic literature.

Since the Cobb-Douglas case has been extensively discussed, for the remainder of this section we will assume that $p \neq 0$. In order to characterize this generalized VES production function, different values of p and ν will be considered within the ranges stated in (4a). If $0 < \nu < 1$, the level of capital per worker must be restricted to obtain non-negative quantities of output, marginal products, and the elasticity of substitution.

3.1 Marginal products

For all $K > 0$ and $L > 0$, let $F_K := \partial F/\partial K$ and $F_L := \partial F/\partial L$. Differentiating (4a) with respect to K and with respect to L gives

$$F_K = \frac{F}{K}\left[\frac{\alpha\nu L^p + (\nu-1)K^p}{L^p + (\nu-1)K^p}\right] \quad \text{and} \quad F_L = \frac{F}{L}\left[\frac{(1-\alpha\nu)L^p}{L^p + (\nu-1)K^p}\right]. \quad (5)$$

[1] A necessary condition for output to be non-negative in (1a) is $\frac{K}{L} \leq \frac{1}{1-\nu}$. Given that $0 < \nu < \frac{1}{\alpha}$, it is sufficient to assume (3).

Both F_K and F_L are nonnegative if the following inequality holds

$$\alpha\nu L^p + (\nu - 1)K^p \geq 0, \tag{6}$$

which is automatically satisfied for $1 \leq \nu < 1/\alpha$. In fact, for these parameter values, marginal products are strictly positive in the interior of $\mathbb{R}^2_+$. If $0 < \nu < 1$, condition (6) implies that

$$\frac{K}{L} \leq \left(\frac{\alpha\nu}{1-\nu}\right)^{1/p}. \tag{7}$$

Now there are two cases to consider: if $0 < p < 1$, then (7) is an upper bound on capital per worker as before; but for $-\infty < p < 0$, the inequality is reversed, so (7) imposes a strictly positive *lower bound* on capital per worker. These bounds are discussed in detail below.

For the sake of consistency, the same notation is used for the second derivatives of F, therefore

$$F_{KK} := \frac{\partial^2 F}{\partial K^2}, \quad F_{LL} := \frac{\partial^2 F}{\partial L^2}, \quad F_{KL} := \frac{\partial^2 F}{\partial L \partial K}, \quad \text{and} \quad F_{LK} := \frac{\partial^2 F}{\partial K \partial L}.$$

Given that F is continuously differentiable, second partial derivatives exist and $F_{KL} = F_{LK}$ for all $K > 0$ and $L > 0$. Hence, the Hessian matrix of F can be completely described by

$$\begin{aligned}
F_{KK} &= -\frac{F_K}{K}\left[\frac{(1-\alpha\nu)L^p}{\alpha\nu L^p + (\nu-1)K^p}\right]\left[\frac{\alpha\nu L^p + (1-p)(\nu-1)K^p}{L^p + (\nu-1)K^p}\right], \\
F_{LL} &= -\frac{F_L}{L}\left[\frac{\alpha\nu L^p + (1-p)(\nu-1)K^p}{L^p + (\nu-1)K^p}\right], \\
F_{KL} = F_{LK} &= \frac{F_L}{K}\left[\frac{\alpha\nu L^p + (1-p)(\nu-1)K^p}{L^p + (\nu-1)K^p}\right].
\end{aligned} \tag{8}$$

A sufficient condition for the concavity of F is a negative semi-definite Hessian matrix, which holds for all $K, L > 0$ if $1 \leq \nu < 1/\alpha$. In the case that $0 < \nu < 1$, the following restriction must be imposed

$$\frac{K}{L} \leq \left[\frac{\alpha\nu}{(1-p)(1-\nu)}\right]^{1/p}. \tag{9}$$

An identical condition arises from the nonnegativity of the elasticity of substitution. Hence, there is no distinction between concavity of the production function and factor substitution in this model.

3.2 Elasticity of substitution

Computing the elasticity of substitution of generalized functional forms can be quite cumbersome using the standard definition based on the (log) derivative of the marginal rate of technical substitution. Here the assumption of constant returns to scale comes to the rescue of σ with a simplified formula. Given a production function F that satisfies usual neoclassical properties and is homogeneous of degree one, the elasticity of substitution reduces to

$$\sigma = \left(\frac{KF_{KK}}{F_K} + \frac{LF_{LL}}{F_L}\right)^{-1}. \tag{10}$$

The proof of this result is given in the Appendix. As a result of this shortcut, the elasticity of substitution for the GVES production function, denoted by σ_p, can be obtained substituting (5) and (8) into (10), which yields

$$\sigma_p = \frac{\alpha\nu L^p + (\nu - 1)K^p}{\alpha\nu L^p + (1-p)(\nu - 1)K^p}. \tag{11}$$

This expression is valid for all $K, L > 0$ if $1 < \nu < 1/\alpha$. In the other case, assuming that (6) holds, the denominator must be strictly positive for the elasticity to be well-defined. This is achieved by (9), but with strict inequality.

3.3 Non-negativity and monotonicity restrictions

It is clear from our previous discussion that we must consider the following parameter configuration for ν and p, that will be identified throughout the paper as cases: (A) $1 < \nu < 1/\alpha$, and (B) $0 < \nu < 1$, with two subcases each: (1) $0 < p < 1$, and (2) $-\infty < p < 0$.

Given that all non-negativity constraints are satisfied for (A1) and (A2), the GVES production function in these two cases is

$$F(K,L) := \begin{cases} K^{\alpha\nu}\left[L^p + (\nu-1)\,K^p\right]^{\frac{1-\alpha\nu}{p}} & \text{if } K > 0 \text{ and } L > 0, \\ 0 & \text{if } K = 0 \text{ or } L = 0. \end{cases} \tag{12}$$

The second line guarantees that $F(K,L)$ is well defined at the origin for all values of p.

In case (B1), the capital-labor ratio is bounded above. This upper bound is obtained from conditions (7)-(9), and defined as the *smallest positive upper bound*

$$\bar{k}_p := \min\left\{\left(\frac{1}{1-\nu}\right)^{\frac{1}{p}}, \left(\frac{\alpha\nu}{1-\nu}\right)^{\frac{1}{p}}, \left[\frac{\alpha\nu}{(1-p)(1-\nu)}\right]^{\frac{1}{p}}\right\}. \tag{13}$$

Given that $0 < \alpha\nu < 1$ and $0 < p < 1$, the upper bound on capital per worker is then

$$\bar{k}_p = \left(\frac{\alpha\nu}{1-\nu}\right)^{\frac{1}{p}}. \tag{14}$$

A different picture emerges for (B2), since now (13) defines the *largest nonnegative lower bound* on K/L, denoted as $\underline{k}_p$, and given by

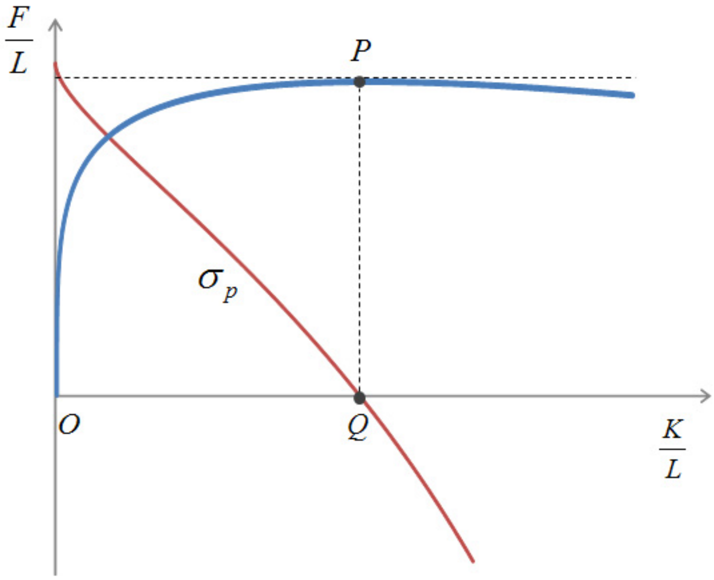
$$\underline{k}_p = \left[\frac{\alpha\nu}{(1-p)(1-\nu)}\right]^{\frac{1}{p}}. \tag{15}$$

Now we are in condition to give a precise definition of the generalized production function over a restricted domain when $0 < \nu < 1$. For case (B1),

$$F(K,L) := \begin{cases} 0 & K = 0 \text{ or } L = 0, \\ K^{\alpha\nu}\left[L^p + (\nu-1)\,K^p\right]^{\frac{1-\alpha\nu}{p}} & K > 0,\ L > 0,\ 0 < \frac{K}{L} \le \bar{k}_p, \end{cases} \tag{16}$$

where $\bar{k}_p$ is defined by (14). This is depicted in Figure 1. Note that it would be inefficient to use more capital than $\bar{k}_p$ (point Q in the graph), because marginal returns become negative beyond that level. Solow (1957) termed this “capital-saturation,” a phenomenon that never arises for standard functional forms, at least in terms of gross output.[2] The production function developed here is able to show capital-saturation as a special case, without having to resort to *ad hoc* assumptions on technology.

Figure 1: Capital saturation



The remaining case (B2), appropriately defined, implies a production set that is not convex over the entire domain (see Figure 2). Non-convexities are pervasive in modern production theory due to the presence of fixed costs of investment, externalities, information asymmetries, market failures, etc. The corresponding definition is then

$$F(K,L) := \begin{cases} 0 & L>0,\ 0 \le \frac{K}{L} < \underline{k}_p \text{ or } L=0, \\ K^{\alpha\nu}\left[L^p + (\nu-1)\,K^p\right]^{\frac{1-\alpha\nu}{p}} & K>0,\ L>0,\ \underline{k}_p \le \frac{K}{L} < \infty, \end{cases} \tag{17}$$

where $\underline{k}_p$ is given by (15).

3.4 Characterization of the elasticity of substitution

As mentioned earlier, the elasticity of substitution varies monotonically with k but needs not be linear. In fact, monotonicity depends only on ν. For certain combinations of the parameters it is possible to identify regions of convexity and concavity. Curvature depends mainly on p, but in a rather complicated way. This is shown in the following proposition.

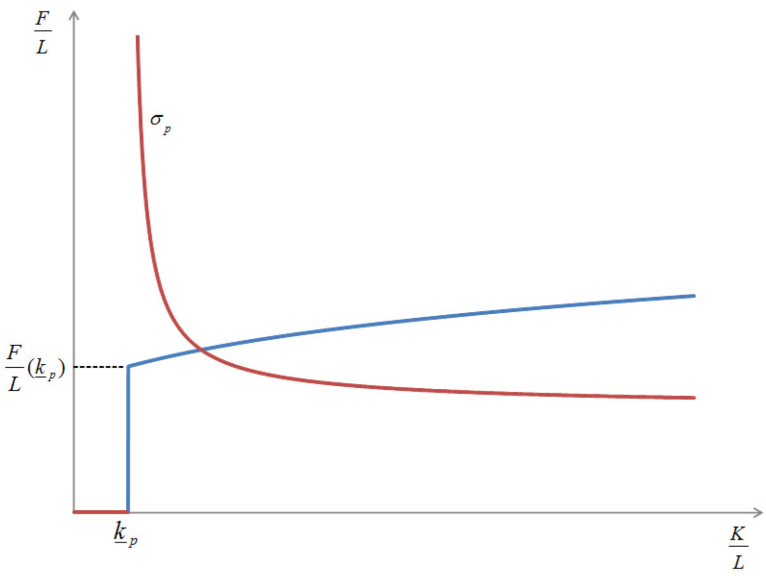
Proposition 3.1. *Let* $F(K,L)$ *be a production function defined by* (12), (16) *or* (17), *according to the parameter values described therein. Let* σ_p *be the elasticity of substitution associated with* F, *given in* (11).

(i) *If* $0<\nu<1$, *the elasticity of substitution is strictly decreasing on the relevant domain;*

(ii) *If* $\nu=1$, *then the elasticity of substitution is constant on* $(0,\infty)$*;*

[2] Capital saturation in terms of net output occurs when the marginal product of capital is lower than the depreciation rate.

Figure 2: Non-convex technology



(iii) *If* $1 < \nu < 1/\alpha$, *then the elasticity of substitution is strictly increasing on* $(0, \infty)$. *Moreover,*

(a) *if* $-1 \leq p < 1$, *then* σ_p *is concave on* $(0, \infty)$;

(b) *if* $-\infty < p < -1$, *for each* p *there is a* $\tilde{k} > 0$ *such that* σ_p *is convex on* $(0, \tilde{k})$ *and concave on* $(\tilde{k}, \infty)$.

Proof. First, rewrite (11) in terms of the capital-labor ratio, so

$$\sigma_p = \frac{\alpha\nu + (\nu - 1)\left(\frac{K}{L}\right)^p}{\alpha\nu + (1-p)(\nu - 1)\left(\frac{K}{L}\right)^p},$$

and differentiate this with respect to K/L on the corresponding domain to obtain

$$\frac{d\sigma_p}{d\left(\frac{K}{L}\right)} = \frac{\alpha\nu(\nu - 1)\,p^2\left(\frac{K}{L}\right)^{p-1}}{\left[\alpha\nu + (1-p)(\nu - 1)\left(\frac{K}{L}\right)^p\right]^2}. \tag{18}$$

The denominator of (18) is strictly positive and the sign of the numerator (and the derivative) is determined by the sign of $(\nu - 1)$. This proves the monotonicity of σ_p in cases (i)-(iii).

For the convexity of the elasticity of substitution in case (iii), differentiation of (18) gives

$$\frac{d^2\sigma_p}{d\left(\frac{K}{L}\right)^2} = -\frac{\alpha\nu(\nu - 1)p^2\left(\frac{K}{L}\right)^{p-2}\left[\alpha\nu + (1+p)(\nu - 1)\left(\frac{K}{L}\right)^p\right]}{\left[\alpha\nu + (1-p)(\nu - 1)\left(\frac{K}{L}\right)^p\right]^3}.$$

By hypothesis, the sign of this second derivative depends on the sign of the expression in square brackets on the numerator, that is,

$$\operatorname{sign}\left[\alpha\nu + (1+p)(\nu - 1)\left(\frac{K}{L}\right)^p\right]. \tag{19}$$

If $-1 \le p < 1$, then (19) is strictly positive for all $K/L > 0$. It follows that the second derivative is negative and σ_p is concave on $(0, \infty)$, as stated in (iii)-(a).

For (iii)-(b), consider (19) as a function of $k := K/L$ for some $-\infty < p < -1$. Denote this new function as $S_p(k)$. It is clear that S_p is continuous and strictly increasing on $(0, \infty)$. Moreover, since $\lim_{k \to 0^+} S_p(k) = -\infty$ and $\lim_{k \to \infty} S_p(k) = \alpha\nu$, there exists a $\tilde{k} > 0$ satisfying $S_p(\tilde{k}) = 0$. Strict monotonicity implies that $\tilde{k}$ is unique and

$$
\begin{aligned}
S_p(k) < 0 \quad &\text{for all} \quad 0 < k < \tilde{k}; \\
S_p(k) > 0 \quad &\text{for all} \quad \tilde{k} < k < \infty.
\end{aligned}
$$

Combining these inequalities with (19) and evaluating them on the second derivative of σ_p yields (iii)-(b). Hence the proof is complete. ■

In several applications knowing whether σ is greater, equal or less than one can be crucial. A simple but appealing feature of the GVES specification developed here is that such distinction depends only on the signs of $(\nu - 1)$ and p.

Lemma 3.2. $\operatorname{sign}(\sigma_p - 1) = \operatorname{sign}(\nu - 1) \operatorname{sign}(p)$.

Proof. Suppose that $(\nu - 1)$ and p take nonzero values. From (11), it is clear that $\sigma_p > 1$ if and only if $\alpha\nu + (\nu - 1)\left(\frac{K}{L}\right)^p > \alpha\nu + (1-p)(\nu - 1)\left(\frac{K}{L}\right)^p$. There are two mutually exclusive combinations of p and ν that satisfy this condition, which are: (i) $p > 0$ and $1 < \nu < 1/\alpha$, or (ii) $p < 0$ and $0 < \nu < 1$. In order to have $\sigma_p < 1$, the inequalities for p in (i) and (ii) are reversed, while the others remain unchanged. Finally, if either $\nu = 1$ or $p = 0$, then $\sigma_p = 1$. Combining all cases together gives the desired result. ■

To close this section, there is a novel – and rather curious – result from this specification. Taking the limit as $p \to -\infty$ yields the following mixed Cobb-Douglas-Leontief technology:

$$F(K, L) = A \cdot \min\left\{K^{\alpha\nu} L^{1-\alpha\nu}, K\right\} \qquad K, L \ge 0.$$

Isoquants are not L-shaped, as in the pure Leontief case, given that the vertical portion of an isoquant now behaves as Cobb-Douglas: it is convex to the origin and has an elasticity of substitution equal to one for all $K/L > 1$. Hence, the GVES specification also contains a Leontief-type technology as a limiting case, which is absent in the canonical VES. Details are shown in the appendix.

4 An application: the Solow-Swan growth model

In this section, the GVES function is embedded into a growth model with neoclassical properties, as developed by Solow (1956) and Swan (1956). For this, consider capital per capita $k := K/L$ is a function of time t. The economy has a constant savings rate $0 < s < 1$ and capital depreciates at a rate given by $\delta > 0$. Population (and the labor force) grows at a constant rate $n > 0$. The following notation is introduced for the derivative of k with respect to time, $\dot{k} := dk/dt$.

Let f be a C^2, strictly increasing and concave production function. Dynamics are built on a basic macroeconomic identity (in per capita terms) which states that for all $t \ge 0$ and $k \ge 0$, gross investment in physical capital equals savings

$$\dot{k}(t) + (n + \delta)\, k(t) \equiv s f\left(k(t)\right),$$

with initial condition $k(0) = k_0 \geq 0$.

In some cases we can take advantage of an implicit restriction on the domain of k. Given that output must be divided into consumption and savings and consumption is nonnegative, there is a maximum sustainable capital stock k_m which solves the following equation

$$f(k_m) - (n + \delta)\, k_m = 0.$$

By the strict concavity of f on $[0, \infty)$ and some additional regularity conditions (e.g. the Inada conditions), there exists a unique $k_m > 0$. This means that there is no loss of generality by assuming that k lies in the interval $[0, k_m]$.

The Solow-Swan growth model can be described as a nonlinear initial value problem (IVP)

$$\begin{cases} \dot{k}(t) = G(k(t)) \\ k(0) = k_0 \geq 0, \end{cases} \tag{20}$$

where $k(t)$ is a path on $[0, \infty)$ and $G(k(t)) := sf\,(k(t)) - (n + \delta)\, k(t)$, for all $k, t \geq 0$. Given that the problem is autonomous, from now on the dependency of the variables on t will be dropped. A solution to this problem exists if G is continuous (Peano's existence theorem).

From the homogeneity assumption, the GVES production function is easily transformed in per capita terms. Let $0 < \alpha < 1$, $1 < \nu < 1/\alpha$ and $p \in (-\infty, 1)\backslash\{0\}$. Output per worker f is described by a function $f : [0, \infty) \to [0, \infty)$, which is defined as

$$f(k) := \begin{cases} Ak^{\alpha\nu}\left[1 + (\nu - 1)k^p\right]^{\frac{1-\alpha\nu}{p}} & \text{if } k > 0, \\ 0 & \text{if } k = 0. \end{cases} \tag{21}$$

If an interior solution of (20) with f given by (21) exists, it must satisfy the following equality

$$k^* = \left[k^{-p} + (\nu - 1)\right]^{-\frac{1}{p}},$$

where

$$k^* := \left(\frac{sA}{n + \delta}\right)^{\frac{1}{1-\alpha\nu}}.$$

In general a solution will be denoted by $\hat{k}_p$. It is shown below that there are two possibilities: either $\hat{k}_p = 0$ or $\hat{k}_p > 0$ is given by

$$\hat{k}_p = \left[(k^*)^{-p} - (\nu - 1)\right]^{-\frac{1}{p}}, \tag{22}$$

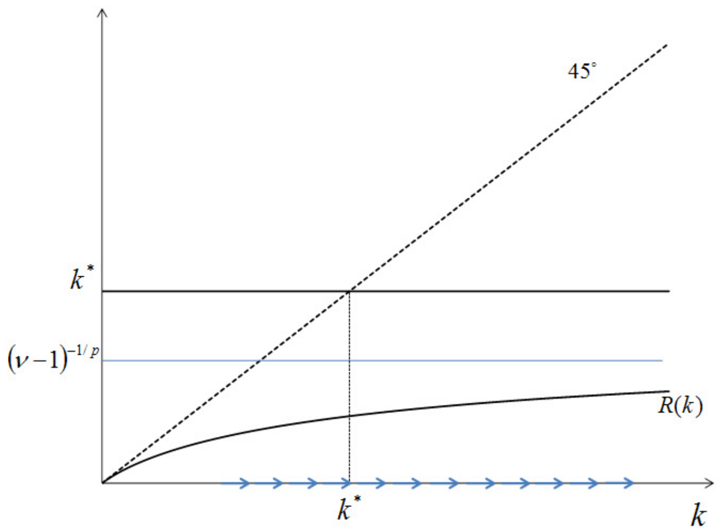
provided that the terms in square brackets is strictly positive, which is equivalent to

$$k^* < (\nu - 1)^{-\frac{1}{p}}. \tag{23}$$

For comparison purposes, the corresponding solution for the canonical VES will be denoted as $\hat{k}_1$. If $\nu = 1$, e.g., the standard Cobb-Douglas case, it is well know that a unique stationary point exists and is equal to

$$k_{\text{CD}} = \left(\frac{sA}{n + \delta}\right)^{\frac{1}{1-\alpha}}.$$

Figure 3: Case A1(a)



Note that (22) can be easily rewritten in terms of k_{CD}

$$\hat{k}_p = \frac{1}{(k_{CD})^{-\frac{1-\alpha}{1-\alpha\nu}} - (\nu - 1)}.$$

Moreover, typically, if $\nu > 1$ then $k^* > k_{CD}$.[3]

In order to characterize the solutions to the IVP (20) with f given by (21), let

$$R(k) := \left[k^{-p} + (\nu - 1)\right]^{-\frac{1}{p}} \quad \text{for all } k > 0.$$

An interior equilibrium point is determined by the intersection of $R(k)$ and k^*. Otherwise, $\hat{k}_p = 0$ holds. By taking the first derivative of $R(k)$, we can see that this curve is strictly increasing on $(0, \infty)$, i.e.,

$$R'(k) = [1 + (\nu - 1)k^p]^{-\frac{1}{p}-1} > 0.$$

Taking the second derivative gives

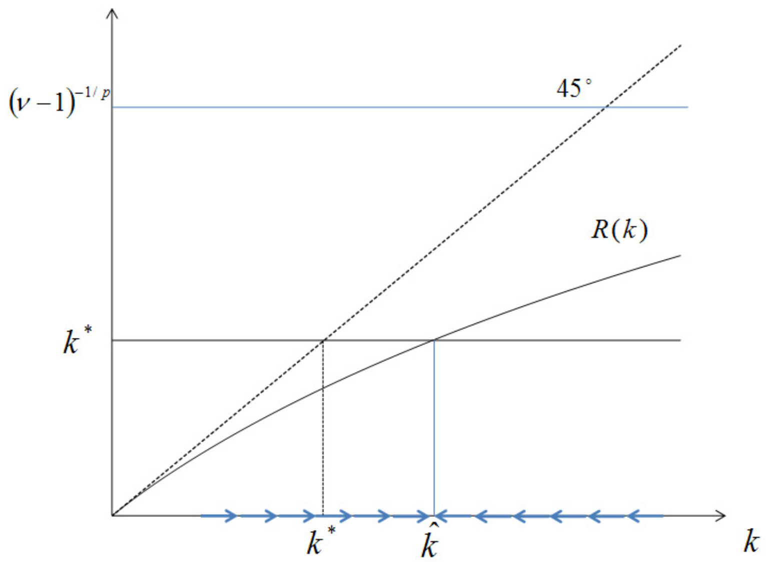
$$R''(k) = -(1 + p)(\nu - 1)\,[1 + (\nu - 1)k^p]^{-\frac{1}{p}-2}\, k^{p-1},$$

so the sign of R'' is determined by $(p + 1)$. In particular, for all $k \in (0, \infty)$, $R(k)$ is concave (resp. convex) provided that $-1 \leq p < 1$ (resp. $-\infty < p \leq -1$). Also, the asymptotic properties of R are useful to describe the dynamics of the model, so let $R(0) := \lim_{k\to 0^+} R(k)$ and $R(\infty) := \lim_{k\to\infty} R(k)$, whenever these limits exist and similarly for R' and R''.

In case (A1), we have $R(0) = 0$, $R(\infty) = (\nu - 1)^{-1/p} > 0$, $R'(0) = 1$ and $R'(\infty) = 0$, which implies that

[3] In fact, this holds true for $sA/(n + \delta) > 1$ and $\nu > 1$. For example, if we assign "typical" values to the parameters involved, e.g., $\alpha = 0.3$, $s = 0.2$, $A = 1$, $n = 0.015$, $\delta = 0.05$, and choose $\nu = 1.2$, then $k^* = 5.8$ and $k_{\text{CD}} = 5$.

Figure 4: Case A1(b)



(a) if $k^* \geq (\nu-1)^{-1/p}$, then $R(k) < k^*$ for all k, so the only stationary equilibrium is the trivial solution $\hat{k}_p = 0$. But this equilibrium is unstable. This is a case of *perpetual growth*, also known in the literature as endogenous growth; for any $k_0 > 0$, capital accumulation lasts forever. Note that this situation is most likely when ν takes relatively high values.

(b) $k^* < (\nu-1)^{-1/p}$, i.e., if (23) holds, there is a steady state equilibrium $\hat{k}_p > 0$ which is global and asymptotically stable. If $(v-1) > 1$ then $k^* < \hat{k}_1 < \hat{k}_p$ for all $p \in (0,1]$. Both cases are depicted in Figures 3 and 4.

Case (A2) implies that $R(0) = (\nu-1)^{-1/p}$, $R(\infty) = \infty$, $R'(0) = \infty$, $R'(\infty) = 1$ and has also two equilibria.

(a) $k^* > (\nu-1)^{-1/p}$, the equilibrium is given by $\hat{k}_p$. If $p = 1$, the intersection must occur below the 45 degree line so the order is unambiguous: $\hat{k}_p < k^* < \hat{k}_1$. Figures 5 and 6 show this case for R concave and convex, respectively.

(b) $k^* \leq (\nu-1)^{-1/p}$ the equilibrium $0 = \hat{k}_p < k^* < \hat{k}_1$ which is global and asymptotically stable, so the economy is caught in a *poverty trap*.

In this simple economy the engine of growth is capital accumulation. Anything that fosters capital accumulation implies higher levels of steady state output. Remember that when $\nu > 1$, the elasticity of substitution is strictly increasing, so the positive effect of an increase in k is reinforced by the fact that capital can be substituted for labor more easily. This second effect is absent in CES technologies because an economy does not have an advantage in terms of factor substitution by becoming more capital intensive. In the opposite case $\nu < 1$, complementary effects between capital and labor are stronger than, for instance, a Cobb-Douglas technology. Consequently, factor substitution becomes more difficult as k increases.

Figure 5: Case A2(a)

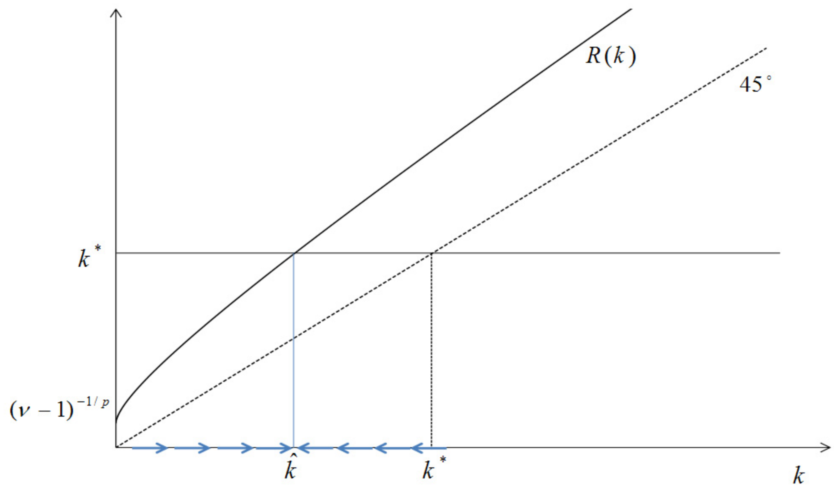
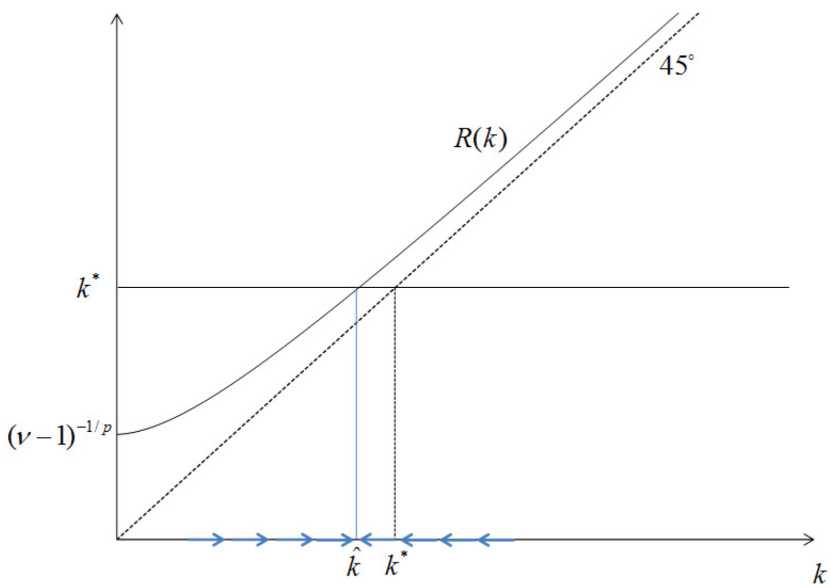


Figure 6: Case A2(b)



A brief comment about cases (B1) and (B2), which have been omitted from this analysis, before ending the section. The presence of capital saturation in (B1) probably makes this case less suitable for a model of economic growth as simple as the one studied here, but it could be more useful if the sources of saturation are explicitly modeled. Non-convexities in (B2) lead to poverty traps for relatively high values of n and δ (or low values of s and A). In the opposite situation, there may be up to two equilibria: one low with $\hat{k}_p > \underline{k}_p$ which is stable, and a high one with $\hat{k}_p < k_m$ which is unstable.

5 Conclusions

This paper develops a generalization of a class of variable elasticity of substitution production functions introduced by Revankar (1971). The generalized function includes as special cases the more representative cases of the CES family that were absent from the original formulation. A complete characterization of the new class is provided, together with an application to the neoclassical growth model.

Appendix

Proof that $p \to 0$ yields VES. We want to find the limit as $p \to 0$ of

$$Y = AK^{\alpha\nu}\Big[L^p + (\nu-1)K^p\Big]^{\frac{1-\alpha\nu}{p}}, \qquad K > 0,\ L > 0.$$

For that, take logs on both members of the above equality

$$\ln Y = \ln A + \alpha\nu \ln K + (1-\alpha\nu)\frac{\ln\left[L^p + (\nu-1)K^p\right]}{p}, \qquad K > 0,\ L > 0,$$

and take the limit as $p \to 0$

$$\lim_{p\to 0} \ln Y = \ln A + \alpha\nu \ln K + (1-\alpha\nu)\lim_{p\to 0}\frac{\ln\left[L^p + (\nu-1)K^p\right]}{p}.$$

Apply L'Hôpital's rule on the last term of the right-hand side,

$$\lim_{p\to 0}\frac{\left[L^p \ln L + (\nu-1)K^p \ln K\right]}{L^p + (\nu-1)K^p} = \frac{\ln L + (\nu-1)\ln K}{\nu}.$$

Hence,

$$\begin{aligned}\lim_{p\to 0} \ln Y &= \ln A + \alpha\nu \ln K + \frac{1-\alpha\nu}{\nu}\left[\ln L + (\nu-1)\ln K\right] \\ &= \ln A + \left(\frac{\nu-1+\alpha\nu}{\nu}\right)\ln K + \left(\frac{1-\alpha\nu}{\nu}\right)\ln L.\end{aligned}$$

Define $\beta := (\nu - 1 + \alpha\nu)/\nu$, and the result follows. ■

Proof of $p \to -\infty$. The proof follows a similar reasoning than the previous one, but there are two cases to consider. First assume that $K > L$. From (4a),

$$Y = AK^{\alpha\nu}L\left[1 + (\nu-1)\left(K/L\right)^p\right]^{\frac{1-\alpha\nu}{p}}.$$

Take logs on this expression to obtain

$$\ln Y = \ln A + \alpha\nu \ln K + (1-\alpha\nu)\ln L + (1-\alpha\nu)\ln\left[1+(\nu-1)(K/L)^p\right]/p,$$

and the limit as $p \to -\infty$,

$$\lim_{p\to-\infty} \ln Y = \ln A + \alpha\nu \ln K + (1-\alpha\nu)\ln L + (1-\alpha\nu)\lim_{p\to-\infty}\ln\left[1+(\nu-1)(K/L)^p\right]/p.$$

Apply L'Hôpital's rule to the last term as before, which yields

$$\lim_{p\to-\infty} \frac{(\nu-1)(K/L)^p \ln(K/L)}{1+(\nu-1)(K/L)^p} = 0.$$

Therefore, $Y = AK^{\alpha\nu}L^{1-\alpha\nu}$. The case $K < L$ can be handled similarly yielding $Y = AK$. Combining both results imply that

$$Y = A\min\left\{K^{\alpha\nu}L^{1-\alpha\nu}, K\right\},$$

and the proof is complete. ■

Proof of σ for F homogeneous of degree one. Let $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ be a C^2 production function that is strictly increasing in both arguments, concave, and homogeneous of degree one. By Euler's theorem for homogeneous functions,

$$KF_K(K,L) + LF_L(K,L) = F(K,L) \qquad K>0,\ L>0.$$

Differentiating the above expression with respect to K and L

$$\begin{aligned} &KF_{KK}(K,L) + LF_{LK}(K,L) = 0 && K>0,\ L>0,\\ &KF_{KL}(K,L) + LF_{LL}(K,L) = 0 && K>0,\ L>0. \end{aligned}$$

Substitute these expressions in (10) and manipulate the result to obtain

$$\begin{aligned}\sigma &= -\frac{F}{KL\left(\dfrac{F_L F_{KK}}{F_K} + \dfrac{KF_{KK}}{L} + \dfrac{F_K F_{LL}}{F_L} + \dfrac{LF_{LL}}{K}\right)},\\ &= -\frac{F}{\dfrac{KF_{KK}}{F_K}(KF_K + LF_L) + \dfrac{LF_{LL}}{F_L}(KF_K + LF_L)}.\end{aligned}$$

Applying Euler's theorem once again in the denominator implies that

$$\sigma = -\frac{1}{\dfrac{KF_{KK}}{F_K} + \dfrac{LF_{LL}}{F_L}},$$

the desired result. ■

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