Measuring vulnerability to multidimensional poverty with Bayesian network classifiers

MAURICIO GALLARDO*

Abstract: Bayesian network methods have recently gained great popularity in machine learning literature and applications to model uncertainty in complex phenomena that include relationships between multiple random variables. However, these models are not commonly applied in economics and development studies. Here, we introduce the use of Bayesian network classifier models to estimate for each individual in a population the probability of being welfare deprived in one and multiple dimensions. These probabilities are then used for measuring vulnerability to multidimensional poverty in four alternative measurement frameworks. Two of them are already known in the research literature but were estimated before with the traditional Probit and Logit unidimensional models. The other two measurement procedures are novel proposals. Using the household survey and the census data from Chile 2017, we provide an empirical application for assessing the performance of these three different Bayesian network classifiers in terms of their accuracy in predicting multidimensional poverty and also to illustrate the four alternative procedures for measuring vulnerability to multidimensional poverty.

JEL Codes: I32, C51, D81, O54.
Keywords: poverty, vulnerability, Bayesian network, Bayesian classifier, downside risk, social risk, Latin America, Chile.

* Escuela de Ciencias Empresariales, Universidad Católica del Norte (UCN), Larrondo 1281, Coquimbo, Chile, e-mail: megallardo@ucn.cl.

Acknowledgments: This work was supported by the Chilean Fondecyt project No. 11170650 "Measuring Vulnerability to Multidimensional Poverty: Methodology and Applications in Latin America".

1. Introduction

As the COVID-19 pandemic has shown, many people who are not poor can eventually be driven into poverty when they are affected by an adverse shock (World Bank, 2020). In the context of poverty and its vicinity, people are threatened by uncertainty and the risk of being severely disturbed by a diversity of downward shocks both idiosyncratic and covariate. Thus, when a measure of poverty is recorded by statistical agencies, such a measure only informs us of how many people are identified as poor in a specific time interval and the magnitude of their deprivation in one or several welfare indicators. However, this register does not inform us about the uncertainty faced by poor people trying to survive, nor about the risk of falling into poverty that the non-poor in the vicinity of poverty confront. To capture this issue, the concept of vulnerability has been established in the economic literature.

Vulnerability concerns the risk that non-poor people have of falling into poverty and the risk that poor people have of remaining poor (Chaudhury et al., 2002). This is a concept that has been involved in a great intellectual debate till this days (see Gallardo, 2018, for a review). However, it is also a phenomenon that the policy-maker cannot ignore in a global society in which we confront not only idiosyncratic risks affecting individuals in their daily lives but also major covariate risks to which entire communities and the whole of civilization are exposed (e.g., the risks associated with climate change, natural disasters and global pandemics).

Vulnerability differs from observed poverty because it concerns the ex-ante risk of being poor, that is, before the veil of uncertainty has been revealed in a factual realisation of either a poverty or non-poverty state of nature (Calvo, 2015).

On the other hand, the understanding of poverty has evolved recently towards a multidimensional perspective (e.g., Anand and Sen, 1997; Mukherjee, 2001; Tsui, 2002; Atkinson, 2003; Bourguignon and Chakravarty, 2003; Chakravarty and Silber, 2008; Asselin, 2009; Alkire and Foster, 2011; Alkire and Santos, 2014; Belhadj and Limam, 2012; Bossert et al., 2013); therefore, empirical research on vulnerability has also moved towards a much more complex concept of vulnerability to multidimensional poverty (VMP). Several measurement approaches have already been proposed in this direction (Calvo, 2008; Abraham and Kavi, 2008; Feeny and McDonald, 2016; Gallardo, 2020). Two of these approaches are based on probability estimations. The first one (Feeny and McDonald, 2016) measures vulnerability as the probability of being multidimensionally poor, while the second one (Gallardo, 2020) estimates the probability of being deprived in each welfare dimension separately and then builds the joint risk following the counting method of Alkire and Foster (2011) (AF method hereafter). The former approach has the advantage of parsimony since it estimates only one probability through a heteroscedastic Probit model, regardless of the specific dimensions in which such multidimensional risk is generated. However, this estimation strategy does not account for the fact that the probability of being multidimensionally poor is determined in different ways conditional to deprivation experienced in different dimensions of well-being. That is, the probability of being multidimensionally poor differs not only according to the deprivation score a person has but also according to the composition of the deprivation set that this score involves.

Regarding the Gallardo (2020) strategy, although it considers the probability of deprivation in each specific welfare dimension, has the disadvantage of a lack of parsimony. When the probability of being deprived in each welfare indicator is estimated separately using Logit or Probit models in each welfare dimension, a great richness of information is obtained. However, bearing in mind that each separate estimate is not free of estimation error, there is a risk that these errors will be then amplified at the aggregation stage.

In this paper, we contribute to the research literature in two directions. First, we introduce Bayesian classifier strategies to estimate the models proposed by Feeny and McDonald (2016) (FM hereafter) and Gallardo (2020) (GM hereafter) instead of using Probit or Logit models as they did. Second, we introduce two new proposals for measuring vulnerability to multidimensional poverty: the first one is a variant of the FM measure in which we also introduce the downside mean-semideviation as the risk parameter; the second one is a new procedure that estimates both the probability of being multidimensionally poor as well as the probability of being deprived in each welfare dimension simultaneously by using a multidimensional Bayesian network classifier (MBC) model in two levels. This new model, at a first level, provides estimates of the probability of being multidimensionally poor by directly using the information contained in the observed deprivations, and then, at a second level, the information on the household and community characteristics is used to estimate the probabilities of being deprived in each welfare dimension.

By introducing the MBC estimation strategies, we push the literature a bit towards the use of properly multidimensional methods to face a problem that is multidimensional in nature, instead of transforming the problem into one or several one-dimensional problems as was done in the FM and GM approaches. Notice that these two previous approaches correspond to what in the multi-label classifier literature (Tsoumakas et al, 2009; Gil Begue et al., 2020) is called a *problem transformation method*, i.e., a method that faces a multidimensional problem, transforming it into one or several one-dimensional problems. In fact, both previous approaches apply a one-dimensional technique (a Probit or Logit model) to solve a multidimensional estimation. Instead, we introduce here an MBC strategy that belongs to the so called *algorithm adaptation method*, i.e., an algorithm that is adapted to directly obtain a multidimensional solution to a multidimensional classification problem.

The MBC estimation strategy allows us to capture in a unified way the contribution of uncertainty in each welfare dimension as in the GM approach together with a simplified estimation of the probability of being multidimensionally poor as in the FM approach. Moreover, this strategy allows us to obtain a robust estimate of the joint probability of multidimensional poverty status, deprivations and household characteristics.

For comparative purposes we offer an empirical illustration of these alternative VMP-measures and estimation procedures employing the same data and variables used by Gallardo (2020). These data correspond to the National Socioeconomic

Characterisation Survey (CASEN) conducted in Chile in 2017. We also use the data from the Chilean national census of the same year to the municipality variables.

The remainder of the paper is organised as follows: in the next section we present the new measurement methodology, followed by an illustrative empirical example using the same data and the same reference multidimensional poverty indicator (MPI hereafter) as in the GM approach to facilitate comparisons. At the end of the paper, we offer some concluding remarks.

2. Methodology

2.1 The estimation problem

Consider a $N \times M$ random matrix Y of M welfare outcomes for a population of N individuals in a cross-sectional time interval t. Each row in matrix Y is a random vector $\mathrm{y}_i = (y_{i1}, ..., y_{iM})$, for which each entry y_{im} is a random binary variable corresponding to the outcome in the welfare dimension m for the individual i. Each random variable y_{im} is equal to one in the event that the individual i reaches a satisfactory well-being level in dimension m and is equal to zero whenever such a person is deprived in that welfare dimension. According to multidimensional poverty literature, the first case corresponds to the condition of being 'non-deprived' while the second case corresponds to the condition of being 'deprived' in such a dimension. In turn, a specific observed realisation of the matrix Y which we denote as Y is usually known as the achievement matrix in poverty literature (see for instance Alkire at al., 2015). That is, the sample space of Y is the set of all possible achievement matrices than can occur in time t.

Given the outcome Y, for any multidimensional poverty measure ψ, there is an identification rule $\psi(\mathrm{y})$ that according to the observed realisation y_i of random vector y_i, allows us to classify any i individual of the population, $i = 1, 2, 3, ...N$ in two categories: multidimensionally poor and multidimensionally non-poor. Any methodology for measuring multidimensional poverty (see for instance the reviews of Deutsch and Silber, 2005; Aselin, 2009; and Alkire at al., 2015) applies an identification rule as $\psi(\mathrm{y})$. However, this identification rule operates ex-post once the veil of uncertainty has been revealed in a factual realisation of the random matrix Y. In contrast, we are interested in the ex-ante situation, which is composed of the possible states of nature and their probabilities.

Let us place ourselves in this ex-ante situation. Under uncertainty, for each individual i we define a binary random variable y_i^w that is equal to one with probability p_i in the event that person i achieves a satisfactory multidimensional level of well-being conditional on the random vector y_i. That is, p_i is the probability for person i to be classified ex-post as multidimensionally non-poor by the identification rule $\psi\left(\mathrm{y}\right)$. Otherwise y_i^w takes the value zero with probability $1-p_i$ in the event that person i does not achieve a satisfactory multidimensional level of well-being given y_i and she is classified ex-post as poor according to the identification rule $\psi\left(\mathrm{y}\right)$. Pay attention to the fact that the variable y_i^w is determined randomly by the vector y_i.

On the other hand, each random variable y_{im} in vector y_i is conditionally dependent on the arrays $\mathrm{x_i}=\left(x_{i1},...,x_{iH}\right)$ and $\mathrm{z_i}=\left(z_{i1},...,z_{iC}\right)$, which respectively are the random vectors of household and community characteristics to which person i belongs. In addition, we assume all random variables $x_{i1},...,x_{iH};z_{i1},...,z_{iC}$ to be categorical.

Our goal is to estimate for each individual i the probability of being multidimensionally poor/non-poor given their characteristics: $\mathrm{y}_i,\mathrm{x_i}$ and $\mathrm{z_i}$. These probabilities will be used later to obtain the VMP measures. Alternatively, we could be interested in estimating the probabilities of being deprived/non-deprived in each welfare dimension, and then we could also use that information to obtain alternative VMP measurements.

In our framework, we will proceed to estimate the following three alternative functions using three different Bayesian classifiers:

$$h_1:\left(x_{i1},...,x_{iH},z_{i1},...,z_{iC}\right)\rightarrow P\left(y_i^w\middle|\mathrm{x_i},\mathrm{z_i}\right) \tag{1}$$

$$h_2:\left(x_{i1},...,x_{iH},z_{i1},...,z_{iC}\right)\rightarrow P\left(y_{im}\middle|\mathrm{x_i},\mathrm{z_i}\right) \tag{2}$$

$$h_3:\left(y_{i1},...,y_{iM},x_{i1},...,x_{iH},z_{i1},...,z_{iC}\right)\rightarrow\left(P\left(y_i^w\middle|\mathrm{y_i}\right),P\left(y_{im}\middle|\mathrm{x_i},\mathrm{z_i}\right)\right) \tag{3}$$

Function h_1 is in the spirit of the FM VMP measure, according to which, given a set of characteristics $\mathrm{x_i}$ and $\mathrm{z_i}$, we obtain the conditional probability of being multidimensionally poor/non-poor: $P\left(y_i^w\middle|\mathrm{x_i},\mathrm{z_i}\right)$. Function h_2 is in the spirit of the GM proposal, according to which, given a set of characteristics $\mathrm{x_i}$ and $\mathrm{z_i}$, we obtain the conditional probability of being deprived/non-deprived in each welfare dimension m: $P\left(y_{im}\middle|\mathrm{x_i},\mathrm{z_i}\right)$. However, instead of estimating such functions through

Probit or Logit models, here we will do it using Bayesian network classifiers. Finally, function h_3 is an approach in a new direction, in which we estimate both the probability of being multidimensionally poor/non-poor conditional on the set of deprivation y_i: $P\left(y_i^w \middle| \mathrm{y}_\mathrm{i}\right)$, as well as the probability of being deprived/non-deprived in each welfare dimension m, conditional on the individual characteristics x_i and z_i: $P\left(y_{im} \middle| \mathrm{x}_\mathrm{i},\mathrm{z}_\mathrm{i}\right)$.

2.2 The Bayesian network classifiers

A Bayesian network (Pearl, 1988; Friedman et al., 1997; Koller and Friedman, 2009; Gil-Begue et al., 2020) is a multivariate probabilistic model over a set of random variables $\mathrm{U}=\{u_1,...,u_I\}$, which is defined through the pair $B(G,\Theta)$. G is a directed acyclic graph (DAG) composed of vertices (also called 'nodes') and arcs. Each node in G represents a random variable of the set U, whereas each arc represents a direct probabilistic dependency relationship. Θ is a vector of parameters $\Theta=\left(\theta_{\boldsymbol{u}_1|\mathbf{pa}_G(\boldsymbol{u}_1)},...,\theta_{\boldsymbol{u}_I|\mathbf{pa}_G(\boldsymbol{u}_I)}\right)$, such that $\theta_{u_i|\mathbf{pa}_G(u_i)}=P\left(u_i \middle| \mathbf{pa}_G\left(u_i\right)\right)$ is the conditional probability of each possible value taken by the random variable u_i given the vector of values taken by the random variables demarcated in G as their parents $\mathbf{pa}_G\left(u_i\right)$ (also called 'ancestors' in the sense of a conditional probabilistic dependency). The Bayesian network defined through the pair $B(G,\Theta)$ represents the following joint probability distribution over the set U according to the structure of G:

$$P_B\left(u_1,...,u_I\right)=\prod_{i=1}^{I} P\left(u_i \middle| \mathbf{pa}_G\left(u_i\right)\right) \tag{4}$$

The conditional dependence relationships established in the topology of G restricts the number of parameters to be estimated to those probabilities conditional to the parents, i.e., 'each variable is independent of its non-descendants in the graph given the state of its parents' (Friedman et al., 1997, p.132).

A Bayesian network classifier, in turn (Friedman et al., 1997; Gil-Begue et al., 2020), is a Bayesian network of specific topology that operates as a function that assigns a labels of one or several class variables $C_1,..,C_S$ to instances $i=1,..,n$, each of which is described by a vector of attributes $\left(a_{i1},...,a_{iJ}\right)$, also called the feature variables.

The simplest Bayesian network classifier is the *Naive Bayes Classifier (NBC)*. It consists of a Bayesian network of only one class variable C with $K \geq 2$ labels, and $J \geq 1$ feature variables, all of which are assumed to be conditionally independent. The NBC assigns a label c_k of the class variable C to each instance i given its attributes $(a_{i1},...,a_{iJ})$. The DAG topology of the NBC as well as the topology of other, more complex Bayesian classifier models is restricted to learning the parameters of the conditional probability of each attribute a_j, $j=1,...,J$ given the class label c_k. That is, in the NBC, the class variable C appears in G as a parent of each feature variable a_j. After that, the Bayes rule is applied to compute the posterior probability of each class label c_k given the set of attributes of the instance i: $P_i\left(c_k \left| a_{i1}, a_{i2}, ..., a_{iJ}\right.\right)$. The class label for each instance i is predicted then with the highest posterior probability.

In our framework, we apply three different Bayesian network classifier models to estimate the probabilities given in the functions h_1, h_2 and h_3 defined before. In what follows, we describe these models.

a) Model 1

The first model is the NBC presented in Figure 1. This model generates the probabilities demarcated in function h_1, and, therefore, it could be used to obtain a Bayesian network version estimate of the FM VMP measure. As it was formally defined above, each node (circle) in the Bayesian network graph represents a random variable, whereas each directed arc (arrow) represents the direction of a conditional dependence between two random variables. According to the NBC topology, the class variable y^w in our model is the parent of the feature variables represented by the elements of the random vectors x and z (we omit subscripts for convenience). Moreover, it is assumed that all feature variables between them are conditionally independent. This simplifying assumption, which is characteristic of the NBC, could sound very strong since one should expect it would be common to find both theoretical arguments and empirical evidence in favour of some conditional dependency between some feature variables. However, despite this strong simplified assumption, the NBC has demonstrated very efficient performance in empirical applications in terms of its predictive accuracy (see for instance Rish, 2001; Ting et al., 2011; Doreswamy and Hemanth, 2011; Fauziyyah et al., 2020).

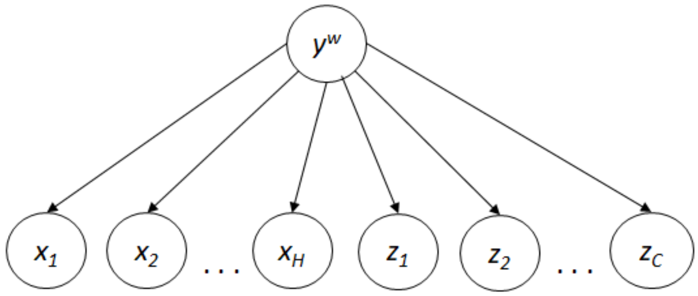

Fig. 1. Naive Bayes classifier to estimate the probabilities defined in function h_1.

To be more explicit in explaining the form in which the NBC operates, notice that the conditional dependencies drawn in the graph are of the form $P\left(x_j \middle| y^w\right)$ and $P\left(z_j \middle| y^w\right)$ for all j. That is, the NBC operates in inverse order of function h_1 since it estimates the probabilities of the feature variables $x_1,...,x_H; z_1,...,z_C$, given the labels of the class variable y^w. However, after that, the probability of interest (the posterior probability) is recovered by applying the Bayes rule as depicted below:

$$P\left(y_i^w \middle| \mathrm{x_i},\mathrm{z_i}\right) = \frac{P\left(y_i^w\right) P\left(\mathrm{x_i},\mathrm{z_i} \middle| y_i^w\right)}{P\left(\mathrm{x_i},\mathrm{z_i}\right)} \tag{5}$$

b) Model 2

Our second Bayesian network classifier model is drawn in Figure 2. This model has a more complex topology. It comprises an MBC (Bielza et al., 2011; Gil-Begue et al., 2020). An MBC is a type of Bayesian network classifier designed to simultaneously solve a multi-dimensional classification problem, i.e., a classification problem that includes multiple class variables. Our MBC model, represented in Figure 2, assigns a classifier vector $\mathrm{y_i} = \left(y_{i1},..., y_{iM}\right)$ to each instance i given the vector of feature variables x and z. This model generates the probabilities defined in function h_2; therefore, it could be used to obtain a Bayesian network estimate of the GM VMP measure.

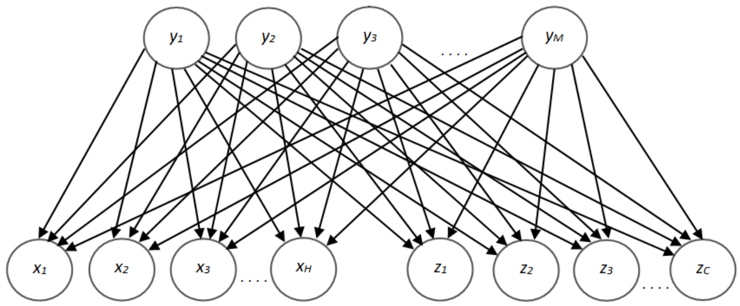

Fig. 2. Multidimensional Bayesian network classifier to estimate the probabilities defined in function h_2.

In a similar way to the above-described NBC model, in the MBC topology, the class variables are drawn as the ancestors (conditioning variables) with respect to the feature variables. That is, the model initially calculates the probabilities of each feature variable conditional on the vector of class variables. Subsequently, the model allows us to infer the probabilities of interest defined in function h_2 (i.e., the posterior probabilities $P\left(y_{im}\middle|\mathrm{x}_\mathrm{i},\mathrm{z}_\mathrm{i}\right)$), also applying the Bayes' rule as follows:

$$P\left(y_{im}\middle|\mathrm{x}_\mathrm{i},\mathrm{z}_\mathrm{i}\right)=\frac{P(y_{im})P\left(\mathrm{x}_\mathrm{i},\mathrm{z}_\mathrm{i}\middle|y_{im}\right)}{P(\mathrm{x}_\mathrm{i},\mathrm{z}_\mathrm{i})} \tag{6}$$

c) Model 3

The third model, represented in Figure 3, is also an MBC. This model only differs from the previous one in incorporating the 'super-class' variable y^w. However, this difference offers the advantage of using all the available information for estimating simultaneously the probabilities of the 'super-class' variable y^w as well as the probabilities of the dimensional class variables $y_{i1},...,y_{iM}$, according to the function defined in h_3.

The idea of using the information available about $y_{i1},...,y_{iM}$ to estimate the probability of y^w is new. The resistance to using such information is understandable, since the effective outcome of y^w is built by definition directly through the rule $\psi\left(\mathrm{y}_i\right)$ from the effective data y_i, which are the realisation of the random vector y_i. So, it might be seen as tautological to use the information from

y_i to generate an outcome that depends on y_i by definition. Note, however, that we are modelling uncertainty and therefore are not operating in the space of compliance of the rule $\psi\left(y_i\right)$ according to y_i, but instead are operating in the space of probabilities of the random vector y_i conditional to y^w. That is, on the one hand, the Bayesian network in Figure 3 provides us with different information than the rule $\psi\left(y_i\right)$. This MBC offers us not just a prediction of the value for the class variable y_i but also its conditional probability. On the other hand, we are using this Bayesian network to predict not the same classification, as the rule $\psi\left(y_i\right)$ already generates by definition, but rather a simulated classification under uncertainty given the specific probabilities of each entity that the Bayesian network itself generates by using the likelihood information contained in the inverse conditionality (i.e., from y^w as ancestor to $y_{i1},...,y_{iM}$ as descendants). The probabilities of function h_3 are then recovered, also applying the Bayes rule as before.

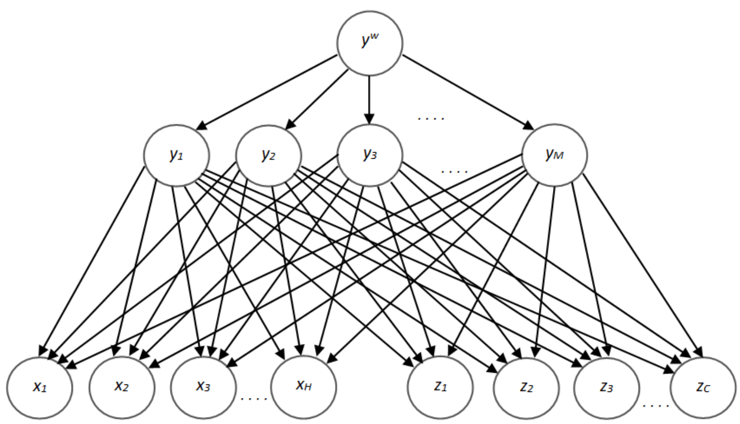

Fig. 3. Multidimensional Bayesian network classifier to estimate the probabilities defined in function h_3.

It should also be added that by directly using the information from $y_{i1},...,y_{iM}$ to infer the probabilities of y^w, Model 3 will be per construction more accurate in predicting the observed outcomes (the realisations of y^w) compared to Models 1 and 2.

2.3 The VMP measures

The probabilities estimated through the Bayesian classifiers described in the previous subsection are subsequently used to estimate four VMP indicators. These indicators are the FM VMP measure, the GM VMP measure and two additional alternative VMP procedures. One of them is similar to that of FM; nonetheless, the criterion to identify vulnerable people is adjusted according to an individual risk parameter defined by the downside mean-semideviation as in Gallardo (2013). The second alternative VMP measure also uses the downside mean-semideviation as the risk parameter. However, it uses the probability estimates obtained through the Bayesian classifiers described in Model 3 instead of using those probabilities obtained in Model 1. In addition, this new model also provides dimensional vulnerability indicators (DVIs) (measures of being vulnerable to each one-dimensional deprivation of those M welfare indicators considered in the multidimensional poverty measurement). These four VMP measures are described in what follows.

2.3.1 Bayesian classifier estimator of the FM VMP measure

According to the FM approach, people vulnerable to multidimensional poverty are identified under the criteria indicated below, which are expressed here in terms of our previously presented framework.

(i) Highly vulnerable: person i is highly vulnerable to multidimensional poverty whenever the probability of being multidimensionally poor for this person fulfils the condition $P\left(y_i^w = 0\right) > 0.5$.

(ii) Relatively vulnerable: person i is relatively vulnerable to multidimensional poverty whenever the probability of being multidimensionally poor for this person fulfils the condition $H < P\left(y_i^w = 0\right) < 0.5$, where H is the headcount ratio of multidimensional poverty in the population.

(iii) Non-vulnerable: person i is non-vulnerable to multidimensional poverty whenever the probability of being multidimensionally poor for this person fulfils the condition $P\left(y_i^w = 0\right) < H$.

Then, in the aggregation phase, three summary VMP measures are obtained in the form of incidence rates for high vulnerability V_{FM}^H, relative vulnerability V_{FM}^R and total vulnerability V_{FM}^{T}. These incidence rates are defined according to the following relation:

$$V_{FM}^{T} = \frac{q_{FM}^{T}}{N} = \frac{q_{FM}^{H} + q_{FM}^{R}}{N} = \frac{q_{FM}^{H}}{N} + \frac{q_{FM}^{R}}{N} = V_{FM}^{H} + V_{FM}^{R} \tag{7}$$

where q_{FM}^{T} is the sum of all vulnerable people, q_{FM}^{H} is the sum of highly vulnerable people, q_{FM}^{R} is the sum of relatively vulnerable people and N is the size of the population.

2.3.2 *Bayesian classifier estimator of the GM VMP measure*

In the GM approach, the people vulnerable to multidimensional poverty are identified in two stages. To understand how the identification rule in the first stage operates, we first define the risk-adjusted mean for person i in the welfare dimension m as follows:

$$\overline{\mu}_{im} = \mu_{im} - \gamma\sigma_{im}^{-} \tag{8}$$

where μ_{im} is the expected value of y_{im}, which is equal to p_{im}, the probability of being non-deprived in this welfare dimension m, given that y_{im} is a Bernoulli distributed random variable equal to one in the event of non-deprivation condition; γ is a risk aversion parameter defined in the interval $(0,1]$; and σ_{im}^{-} is an asymmetric risk parameter, namely the standard downside mean-semideviation.[1] For each variable y_{im}, this risk parameter is defined as follows:

$$\sigma_{im}^{-} = \sqrt{p_{im}^{2}\left(1 - p_{im}\right)} \tag{9}$$

where p_{im} is the probability of being non-deprived in the welfare dimension m.[2] In the first stage, person i is identified as vulnerable in the welfare dimension m whenever his risk-adjusted mean fulfils the condition $\overline{\mu}_{im} \leq z_{im}^{v}$, where z_{im}^{v} is the

[1] This framework is supported on a mean-risk behavior model (see Ogryczak and Ruszczynski, 1999, 2001).

[2] The general definition of the standard downside mean–semideviation is $\sigma_{im}^{-} = \sqrt{E\left\{\min\left[\left(y_{im} - \mu_{im}\right), 0\right]^{2}\right\}}$, but it takes the form expressed in (11) for a Bernoulli random variable.

probability that defines the vulnerability threshold in dimension m. In the empirical application provided by Gallardo (2020) the probability threshold z_{im}^{v} is defined through a receiver operating characteristic (ROC) curve analysis, as was done by Hohberg et al. (2018). However, to be consistent with our estimation framework based on Bayesian network classifiers, we will use here the standard probability threshold of 0.5. We recall that when a Bayesian network classifier is applied, the class label is predicted through the maximum posterior probability and that when we only have two labels (deprived and non-deprived in this case) the threshold to overcome to achieve the highest posterior probability is 0.5.

This threshold also has significant advantages for comparability purposes in contrast to the use an ROC curve rule, first because it allows comparison with other indicators that also use this threshold (e.g., with the FM measure presented above) and second because the use of an ROC curve to determine the vulnerability threshold is a data-dependent endogenous rule (i.e., the threshold will change depending on the data in each temporal or spatial unit), which allows neither inter-temporal comparison nor comparison between geographical units (e.g., countries, regions).

Once people are identified as vulnerable in each welfare dimension, in the second stage, the GM approach proceeds as in the AF method (Alkire and Foster, 2011; Alkire et al., 2015, Chapter 5) to solve the multidimensional identification and aggregation of multidimensional vulnerability. To do that, the vulnerability score for person i is calculated as follows:

$$s_i^v = \sum_{m=1}^{M} w_m g_{im}^0 \tag{10}$$

where g_{im}^0 is an indicator function equal to one when person i is vulnerable in dimension m and equal to zero otherwise; w_m is the weight of the dimension m in the multidimensional poverty measurement, such that $\sum_{m=1}^{M} w_m = 1$. Then, person i is identified as vulnerable to multidimensional poverty whenever the following condition is fulfilled: $s_i^v \geq k$, where k is the multidimensional poverty threshold.

Finally, in the aggregation phase, Gallardo (2020) introduced several VMP indicators; between them, for comparative purposes, we will focus our attention only on the simplest one: the VMP headcount ratio. This indicator is simply the sum of vulnerable people q_G divided by the number of people in the population as defined below:

$$V_G^T = \frac{q_G}{N} \tag{11}$$

2.3.3 An asymmetric variant of the FM VMP measure (FM-A)

The FM VMP measure presented before is an approach based on the concept of vulnerability as expected poverty, which has been the most popular framework for assessing vulnerability to monetary poverty since the beginning of the century (Chaudhuri et al., 2002; Suryahadi and Sumarto, 2003; Christiaensen and Subbarao, 2005). However, some researchers have criticised this approach to vulnerability, which is based only on the expected value, due to its lack of risk sensitivity as variability (see, e.g., Ligon and Schechter, 2002; Hoddinott and Quisumbing, 2003). Even more, this approach has also been criticised due to its lack of sensitivity to downside risk (Gallardo, 2018). In response to this criticism, we introduce here an asymmetric variant of the FM VMP measure that also incorporates the standard downside mean-semideviation as the risk parameter in a similar mean-risk behaviour framework as proposed in Gallardo (2013) for assessing vulnerability to monetary poverty.

Under this approach, for each person i, we obtain the probability of being multidimensionally non-poor p_i directly from the NBC described in Model 1. This probability is in turn equal to the expected value $E\left(y_i^w\right) = \mu_i$, since y_i^w is a Bernoulli random variable equal to one whenever person i is multidimensionally non-poor. In the next step, we calculate the risk-adjusted mean of y_i^w for person i in a similar way as was done in Equation (8) above:

$$\overline{\mu}_i = \mu_i - \gamma\sigma_i^- \tag{12}$$

where now σ_i^- is the standard downside mean-semideviation of y_i^w defined in this case with the probability of being multidimensionally non-poor p_i:

$$\sigma_i^- = \sqrt{p_i^2\left(1 - p_i\right)} \tag{13}$$

Vulnerable people are identified according to the rule that person i is vulnerable to multidimensional poverty whenever her risk-adjusted mean fulfils the condition $\overline{\mu}_i \leq 0.5$. In addition, we can also distinguish between those whose vulnerability is determined by their expected poverty condition and those whose vulnerability is determined by risk. The first ones are those who fulfil the condition $\mu_i \leq 0.5$, while the second ones are those whose expected value of y_i^w is greater than 0.5 but whose risk-adjusted mean does not exceed this threshold (i.e., those who simultaneously fulfil the conditions $\mu_i > 0.5$ and $\overline{\mu}_i \leq 0.5$). Accordingly, we can summarise the vulnerability of the whole population by computing the headcount ratio of total vulnerability. Then, following the taxonomy defined in Günther and Harttgen (2009), we can decompose this incidence rate into two elements corresponding to the headcount ratios of *poverty-induced vulnerability* and *risk-induced vulnerability*. These headcount ratios are defined through the following relation:

$$V_{FM-A}^{T} = \frac{q_{FM-A}^{T}}{N} = \frac{q_{FM-A}^{P} + q_{FM-A}^{R}}{N} = \frac{q_{FM-A}^{P}}{N} + \frac{q_{FM-A}^{R}}{N} = V_{FM-A}^{P} + V_{FM-A}^{R} \quad (14)$$

where V_{FM-A}^{T}, V_{FM-A}^{P} and V_{FM-A}^{R} are the headcount ratios of total vulnerability, poverty-induced vulnerability and risk-induced vulnerability, respectively; q_{FM-A}^{T} is the sum of all vulnerable people; q_{FM-A}^{P} is the sum of those poverty-vulnerable (i.e., vulnerable by expected poverty); q_{FM-A}^{R} is the sum of those risk-vulnerable; and N is the size of the population. We use here the sub-indices $FM-A$ to emphasise that this is an asymmetric variant of the FM VMP measure.

2.3.4 A new approach: the MBC VMP measure

This new approach is based on Model 3, an MBC that provides estimates for each person i of both the probability to be non-deprived in each welfare dimension m, as well as the probability of being multidimensionally non-poor, already denoted as p_{im} and p_i, respectively. The estimates of the probabilities p_{im} allow us to obtain vulnerability measures in each welfare dimension as in the GM approach by using the risk-adjusted mean defined in Equation (8) and the criterion $\overline{\mu}_{im} \leq 0.5$ to identify the vulnerable people in each welfare dimension m. Meanwhile the estimate of the probability p_i allows us to obtain a VMP measure in a similar way as in the FM-A approach described above by using the risk-adjusted mean defined in Equation (12) and the criterion $\overline{\mu}_i \leq 0.5$ to identify the people vulnerable to multidimensional poverty.

In the aggregation phase, we obtain incidence rates of vulnerability in each welfare dimension m, as well as the incidence rate of multidimensional vulnerability, which we denote as $V_{MBC,m}^{T}$ and V_{MBC}^{T}, respectively. Furthermore, as defined similarly in Equation (14), under this approach, it is also possible to carry out the decomposition between *poverty-induced* and *risk-induced* vulnerability both in the unidimensional and multidimensional measurements as defined below:

$$V_{MBC,m}^{T}=\frac{q_{MBC,m}^{T}}{N}=\frac{q_{MBC,m}^{P}+q_{MBC,m}^{R}}{N}=\frac{q_{MBC,m}^{P}}{N}+\frac{q_{MBC,m}^{R}}{N}=V_{MBC,m}^{P}+V_{MBC,m}^{R} \tag{15}$$

$$V_{MBC}^{T}=\frac{q_{MBC}^{T}}{N}=\frac{q_{MBC}^{P}+q_{MBC}^{R}}{N}=\frac{q_{MBC}^{P}}{N}+\frac{q_{MBC}^{R}}{N}=V_{MBC}^{P}+V_{MBC}^{R} \tag{16}$$

where $V_{MBC,m}^{T}$, $V_{MBC,m}^{P}$, $V_{MBC,m}^{R}$ and V_{MBC}^{T}, V_{MBC}^{P}, V_{MBC}^{R} are the headcount ratios of total vulnerability, poverty-induced vulnerability and risk-induced vulnerability for unidimensional and multidimensional vulnerability, respectively; $q_{MBC,m}^{T}$ and q_{MBC}^{T} are the sums of all unidimensional and multidimensional vulnerable people, respectively; $q_{MBC,m}^{P}$ and q_{MBC}^{P} are the sums of those unidimensionally and multidimensionally poverty-vulnerable, respectively; $q_{MBC,m}^{R}$ and q_{MBC}^{R} are the sums of those unidimensionally and multidimensionally risk-vulnerable, respectively; and N is the population size.

In addition to the above, it is also possible to calculate summary measures of multidimensional vulnerability gaps and quadratic multidimensional vulnerability gaps using the Foster-Greer-Thorbecke (FGT) design (Foster et al., 1984). To do this, we first define the multidimensional vulnerability gap for person i as follows:

$$g_i=\begin{cases}\left(\dfrac{z^{v}-\overline{\mu}_i}{z^{v}}\right),\forall\overline{\mu}_i<z^{v}\\ Min\left(\dfrac{z^{v}-\overline{\mu}_i}{z^{v}}\right),\forall\overline{\mu}_i=z^{v}\ .\\ 0,\forall\overline{\mu}_i>z^{v}\end{cases} \tag{17}$$

where z^{v} is the probability that defines the vulnerability threshold, i.e., $z^{v}=0.5$ in our framework. Notice that in the exceptional case where $\overline{\mu}_i=z^{v}$, the minimum observed gap value is assigned. This is consequent with the assumption that the group of vulnerable people is a closed set in the vulnerability threshold. That is, intuitively, we assume that a person who has equal chances to be poor or non-poor when we adjust these chances for a standard downside mean-semideviation must be identified as vulnerable. In other words, we assume there is no certainty in the vulnerability threshold. However, the empirical effect of allocating the value of the minimum observed gap rather than zero in these exceptional cases will be irrelevant in practice given its small magnitude in comparison with the aggregate of all vulnerability gaps.

By aggregating the vulnerability using the FGT design, a more general family of VMP measures could be defined as follows:

$$V_\alpha = \frac{1}{N}\sum_{i=1}^{N} g_i^\alpha, \alpha \geq 0. \tag{18}$$

Here for $\alpha = 0$, we obtain the measure $V_0 = V_{MBC}^T$ defined in Equation (16). For $\alpha = 1$, the expression in Equation (18) is the multidimensional vulnerability gap V_1, while for $\alpha = 2$, this is the squared multidimensional vulnerability gap V_2.

Finally, the decomposition applied to the headcount ratio in Equation (16) could also be extended to a more general decomposition of the V_α family (see also Gallardo, 2013) as follows:

$$\begin{aligned} V_\alpha^T &= V_\alpha^P + V_\alpha^R \\ V_\alpha^P &= \sum_{i=1}^{N} \frac{1}{N} g_i^\alpha I_{\mu_i \leq z^v}, \ \alpha \geqslant 0 \\ V_\alpha^R &= \sum_{i=1}^{N} \frac{1}{N} g_i^\alpha I_{z^v < \mu_i \leq z^v + \gamma\sigma^-}, \alpha \geqslant 0 \end{aligned} \tag{19}$$

where V_α^P and V_α^R are the corresponding measures of poverty-induced and risk-induced vulnerability measures, respectively, while $I_{\mu_i \leqslant z}$ and $I_{z^v < \mu_i \leq z^v + \gamma\sigma^-}$ are binary indicator functions equal to one when the conditions $\mu_i \geq z^v$ and $z^v < \mu_i \leq z^v + \gamma\sigma^-$ are fulfilled respectively for person i and are equal to zero otherwise.

To conclude this sub-section of VMP measurements, we also define two complementary indicators proposed by Gallardo (2020). These indicators are useful when comparing the headcount ratios of multidimensional poverty H and multidimensional vulnerability V_{MBC}^T. They are the vulnerability-to-poverty ratio (VPR) and the over-rate-of-vulnerability headcount ratio (ORV), as defined below:

$$VPR = \frac{V_{MBC}^{T}}{H} \quad (20)$$

$$ORV = V_{MBC}^{T} - H \,. \quad (21)$$

VPR informs us how many vulnerable people there are proportionally per each poor person in a population. Meanwhile, the ORV reports information about the additional proportion of people who are vulnerable to multidimensional poverty over the proportion of people who are multidimensionally poor.

3. Empirical Illustration

For comparative purposes, we illustrate the alternative models described in the previous section, taking the same data and variables used by Gallardo (2020). In this section, we briefly describe the data and the structure of the reference indicator of multidimensional poverty (MPI), following which we present the results of the predictive accuracy of the three Bayesian network classifier models in our framework. Finally, we offer the results of the VMP measurements.

3.1 Reference indicator and data

To measure VMP, a reference MPI is needed. This reference MPI is the specific framework that is used for measuring the observed multidimensional poverty, i.e., it is the specific form taken by the rule $\psi\left(\mathrm{y}_i\right)$. Here, we use the national MPI design of Chile, in which the AF method (Alkire and Foster, 2011) is applied.

The structure of the Chilean MPI is shown in Figure 4. This MPI is composed of 15 welfare indicators (dimensions $y_{i1},...,y_{iM}$ in our framework) that are grouped in five dimensional blocks. The core dimensions are grouped in four blocks: ‘education’, ‘health’, ‘employment and social security’ and ‘housing and environment’. These core dimensions belong to the original Chilean MPI design, which was computed from 2009. Higher weights (22.5% for each block and 7.5% for each indicator) have also been assigned to these core dimensions in the current MPI composition, which was updated in 2015. In this update, an additional dimensional block of ‘networks and social cohesion’ was included. According to the official methodological document of the Chilean MPI (CASEN, 2016), this improvement was introduced with the aim of extending the Chilean MPI to the so-called ‘missing dimensions’, which are related to the social participation usually not considered in other MPI designs around the world. This new dimensional block is weighted lower in the current Chilean MPI (10% as a block and 3.33% by indicator), as is shown in Figure 4.

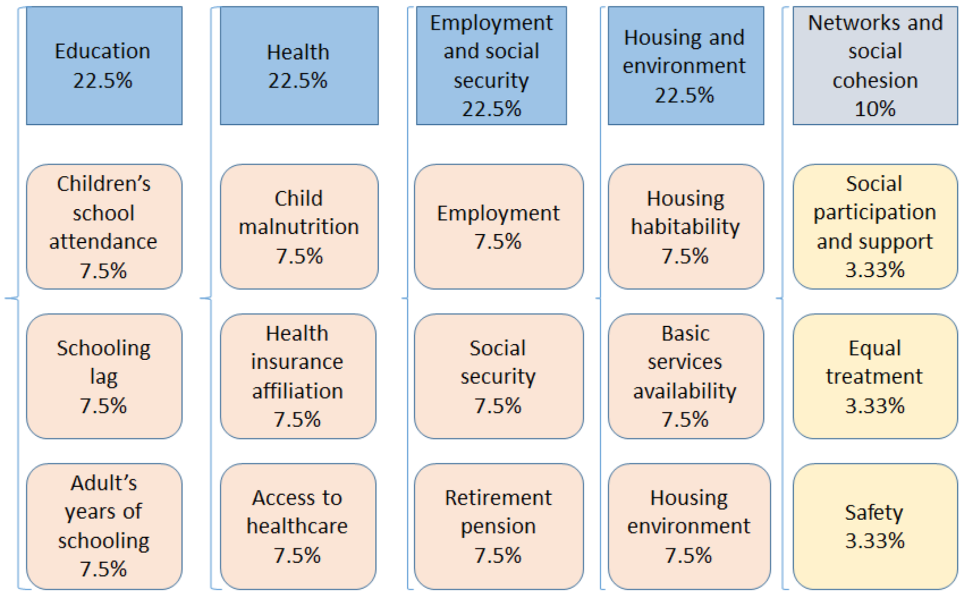

Fig. 4. Dimensional structure and indicators of the Chilean MPI.

In any MPI, for each welfare indicator, there is a deprivation cut-off that defines the threshold beyond which the members of a household must be identified as deprived in that indicator. In order to avoid too many details, we do not indicate here what the cut-offs are for each of the 15 indicators in the Chilean MPI. However, an interested reader can consult them in Table A1 in the Appendix.

It should be added also that both in the process of computing the MPI and in the process of estimating the VMP measures, the identification unit is the household and not the individual. That is, the identification process is performed through the characteristics of the household to which the person belongs and not in reference to the person himself. To this respect, it is argued that the household characteristics are shared by all their members and therefore affect all of them (Santos, 2019). For instance, if a child in a household does not attend school, then all members of this household are going to be affected by this deprivation and must all be identified as deprived in the indicator of school attendance. Likewise, if an elderly person does not have a retirement pension, then all the members of that household will be considered deprived in that welfare indicator. However, although the individuals are identified as multidimensionally poor and vulnerable at the household level, all further computations are performed considering the individual as the statistical unit.[3]

As the Chilean MPI follows the AF method, first, the households are identified as deprived or non-deprived in each welfare indicator depending on whether or not they reach the cut-off point in such indicator. Second, a deprivation score is computed for each household as the weighted sum of deprivations (using the indicator weights in Figure 4). Finally, those people in whose households the deprivation score exceeds a certain threshold k are identified as multidimensionally poor. We apply here a multidimensional threshold $k = 0.225$, which is the official threshold in Chile. However, we also evaluate the performance of the VMP measurement under two alternative k values: 0.15 and 0.3.

The two main summary multidimensional poverty statistics in the AF method are the multidimensional headcount ratio H and the multidimensional poverty index M_0. The former is the proportion of multidimensionally poor people in a population, while the second is defined as $M_0 = H \times A$, where A is the intensity of deprivations among the poor. In other words, A is equal to the average deprivation score among multidimensionally poor people.

The data used in this empirical illustration were taken from the National Socioeconomic Characterisation Survey (CASEN) from 2017. The CASEN is a periodical household survey with national coverage and statistical representativeness of the whole population. The CASEN survey is widely used in Chile to measure both monetary and multidimensional poverty, as well as other socio-economic statistics. It is a survey of complex sample design, stratified by territorial conglomerates, also called ‘segments’. The sample used in this application comprises 67,820 individuals, which, according to the expansion factors, represents a population of 17,060,700 persons in total, or 92% of the total Chilean population as computed in the 2017 census.

Because our application requires that both the household and community feature variables be categorical, a discretisation of them was carried out. Table 1 shows the discretisation implemented with the household feature variables ($x_{i1},...,x_{iH}$ in our framework).

Regarding the community feature variables, they correspond to municipalities in which people live. These variables are taken directly from the 2017 census data. These municipality variables are mean years of education, share of the indigenous population, share of people employed in mining, share of people employed in manufacturing and share of people employed in agriculture. We discretised all these municipality variables by quintile. For details on the descriptive statistics of all

[3] The standard computation involves a double weighting procedure, with one expansion factor from the household to its members (usually the household size) and another from the sample unit to the population.

feature variables used in this empirical application, see Gallardo (2020, Table 2) as we use the same data for this illustrative example.

Table 1. Discretisation of feature household variables. Proportions for each label.

	Categories	Sample (67,820)	Expanded (17,060,700)
Household head years of schooling	Postgraduate	1.6%	1.9%
	Complete Tertiary	17.9%	19.4%
	Complete Secondary	27.6%	30.1%
	Complete primary	13.7%	12.9%
	Incomplete Tertiary	5.1%	5.2%
	Incomplete Secondary	13.5%	13.8%
	Incomplete primary	17.7%	14.7%
	No education	2.9%	2.1%
Household head age	<=25	2.7%	2.4%
	26-35	12.1%	13.6%
	36-45	17.2%	20.2%
	46-55	21.6%	23.8%
	56-65	20.9%	20.1%
	>65	25.6%	19.9%
Household head gender	Man	58.9%	61.2%
	Woman	41.1%	38.8%
Household size	1 person	15.8%	5.0%
	2 persons	25.8%	16.6%
	3 persons	23.3%	23.1%
	4 persons	19.2%	25.4%
	5 persons	9.5%	15.6%
	6 persons	3.7%	7.4%
	7 persons	1.5%	3.3%
	8 persons	0.6%	1.8%
	9 persons or more	0.5%	1.8%

Table 1. (continuation) Discretisation of feature household variables. Proportions for each label.

	Categories	Sample (67,820)	Expanded (17,060,700)
Household head ethnicity	No ethnic group	89.5%	91.7%
	Ethnic group	10.5%	8.3%
Rural location	Urban	81.1%	87.4%
	Rural	19.0%	12.6%
Dependency rate	Zero dependency	24.0%	17.7%
	0 - 0.5	20.8%	26.2%
	0,5 - 1	24.0%	28.2%
	1 - 1.5	4.0%	6.9%
	1.5 - 2	7.1%	8.1%
	2 - 3	2.3%	3.3%
	3 - 4	0.6%	1.0%
	>4	17.2%	8.7%

3.2 Predictive accuracy of the Bayesian network classifiers

To estimate all the Bayesian network models presented in the methodological section, we used the *bnlearn* R-package developed by Scutari (2010). The Bayesian parameter estimation method available in this R-package was applied. We must also clarify that in the estimation of the MBC Models 2 and 3, an arc corresponding to the relationship between the years of adults' schooling in the household and the years of the household heads' schooling was eliminated. This arc has no sense since, in this case, the feature variable is part of the class variable.

To assess the predictive accuracy of the three Bayesian network classifiers, a five-fold cross-validation exercise was performed. We estimated two measures of accuracy. One of them, the *average accuracy* of the dimensional class variables, was used to evaluate performance over the class-dimensional variables only in the MBC Models 2 and 3 as Model 1 does not include class dimensional variables. The other measure used in the cross-validation, which we call here the *overall accuracy*, has the aim of assessing the global performance of each model in predicting the observed values of the main class variable y^w. This strategy of assessing the predictive accuracy of an MBC model through two measures, one of them for the prediction accuracy of the class variables separately and another for the global prediction, is in line with similar assessments that have been implemented in the literature (see, e.g., Bielza et al., 2011; Zaragoza et al., 2011). Accordingly, the average accuracy is defined as follows:

$$\overline{Acc_M} = \frac{1}{M}\sum_{m=1}^{M} Acc_m \tag{22}$$

where Acc_m is the predictive accuracy for the class-dimensional variable m, i.e., the number of correctly classified observations in that dimension divided by the total observations. The overall accuracy is instead just the accuracy in predicting the observed values of y^w, i.e., it is equal to the number of correctly classified observations of the multidimensionally poor and non-poor divided by the size of the population considered in the sample as defined below:

$$Acc_{y^w} = \frac{1}{N}\sum_{i=1}^{N} I_{y^w} \tag{23}$$

where I_{y^w} is an indicator function equal to one if the observed value of y^w is correctly predicted by the model for the individual i and is equal to zero otherwise.

Table 2 presents the performance statistics of the three estimated Bayesian classifier models. The two MBC models perform better in predicting the multidimensional poverty/non-poverty condition with an overall accuracy of 0.84 for Model 2 and 0.96 for Model 3. In contrast, the overall accuracy of Model 1 is the lowest: 0.8. Meanwhile, the average accuracy of the class dimensional variables is quite similar for Models 2 and 3.

Table 2 also shows the accuracy of Models 2 and 3 in predicting the outcomes of each class dimensional variable. We observe that the dimensional indicators that are more difficult to predict are the ‘years of adults’ schooling’ and ‘social security’. For these dimensions, the MBC models present the lowest accuracy. On the other hand, there are some dimensional indicators to which the two MBC models offer very high performance, 0.9 or above, in terms of predictive accuracy. These dimensional indicators are ‘children’s school attendance’, ‘schooling lag’, ‘health insurance affiliation’ and ‘social participation and support’. In general, however, the predictive performance of the two models is high in the dimensional indicators.

Table 2. Predictive accuracy of the Bayesian network classifiers with five-fold cross-validation

Dimension	Model 1	Model 2	Model 3
Overall accuracy	0.80	0.84	0.96
Average Accuracy over Dimensions	--	0.85	0.86
Accuracy by dimensions			
Children's school attendance		0.92	0.92
Schooling lag		0.92	0.92
Years of adults' schooling		0.68	0.73
Child malnutrition		0.89	0.90
Health insurance affiliation		0.90	0.90
Access to healthcare		0.91	0.91
Employment		0.87	0.87
Social security		0.66	0.70
Retirement insurance		0.84	0.85
Habitability		0.78	0.80
Basic services		0.85	0.88
Local environment		0.85	0.86
Equal treatment		0.85	0.85
Safety		0.87	0.87
Social participation and support		0.90	0.90

With the aim of exploring how the performance of the three Bayesian classifiers changes under different MPI specifications, we have also developed a sensitive analysis of their predictive accuracy under different scenarios. Following Gallardo (2020), in addition to the official Chilean MPI, which is our baseline scenario, we also estimated the three models for two alternative multidimensional poverty cut-offs, two alternative weight configurations and two alternative MPI compositions. The two alternative multidimensional poverty cut-offs are $k = 0.15$ and $k = 0.3$. The first one is laxer, which would imply a greater number of vulnerable people, while the other one is more stringent and would imply a lower vulnerability headcount ratio. The two alternative scenarios of weight configurations are justified by public policy reasons (see Gallardo, 2020). On the one hand, given the difficulty of justifying the weight assignment (Decancq and Lugo, 2013), the possibility of equal weights to all dimensions is always of interest as a valid standard alternative.

The equal weight configuration has been implemented in several MPI designs (e.g., Alkire and Santos, 2014; Chowdhury and Squire, 2006). On the other hand, it makes sense to ask ourselves whether the dimensional block of networks and social cohesion should weigh even less than 10% under the argument that the most acute poverty is much more closely linked to the other four dimensional blocks. Thus, the chosen weighting alternatives are equal weights for all dimensions and less weight (6%) for the dimensional block of networks and social cohesion. Regarding the alternative MPI compositions, it is evident that the original design with only four dimensional blocks is relevant, as is the alternative of including the monetary dimension to the current official design. The second alternative has been supported before by other researchers who have included the monetary dimension in their MPI proposals for measuring multidimensional poverty in Latin America (Battiston et al., 2013; Santos and Villatoro, 2018).

Table 3 presents the results of the overall accuracy for all the alternative scenarios in comparison with the baseline official design. As an indicator of dispersion, we offer the range of accuracy for each model in the last column. The average accuracy is not presented here as it does not change for the comparisons between the alternative multidimensional poverty cut-off and weights configuration, whereas between different MPI compositions it only changes slightly.

Table 3. Sensitivity analysis of predictive accuracy of the Bayesian network classifiers for different scenarios. A five-fold cross-validation.

	Different multidimensional poverty cut-offs			
Bayesian classifier	*k=0.225*	*k=0.15*	*k=0.3*	*Range*
Model 1	0.80	0.68	0.92	0.231
Model 2	0.84	0.75	0.84	0.098
Model 3	0.96	0.99	0.98	0.024
	Different weight configurations			
	Official MPI	*Equal weights*	*Lower weights for networks and social cohesion blocks*	*Range*
Model 1	0.80	0.88	0.80	0.084
Model 2	0.84	0.91	0.82	0.092
Model 3	0.96	0.97	0.96	0.009
	Different MPI compositions			
	Official MPI	*Four dimensional blocks MPI*	*Official MPI plus monetary dimension*	*Range*
Model 1	0.80	0.80	0.84	0.044
Model 2	0.84	0.81	0.86	0.048
Model 3	0.96	0.97	0.98	0.013

As we see in Table 3, Model 3 performs better in all scenarios. It is also noteworthy that the overall accuracy between the alternative MPI scenarios is also more stable for Model 3 as it presents the lower range in this performance measurement in all comparisons. Model 1 is the least stable in the face of changes in multidimensional poverty cut-offs, but it performs similarly in stability with respect to Model 2 in the other two comparisons. In general, all three models perform very well in predicting the outcome of the class variable y^w with an overall accuracy of 0.8 or above. The only two cases with overall accuracy less than 0.8 are for Models 1 and 2 in the alternative scenario of $k = 0.15$. This is understandable as it is more difficult to predict poor people when the multidimensional poverty threshold is too low.

3.3 VMP measurement results

As was explained in the methodological section, we used the probabilities provided by the Bayesian network classifiers to estimate four different VMP measurements. The FM-A, GM and MBC measures were estimated for different gamma values in the range from $\gamma = 0.1$ to $\gamma = 1$. The details of these estimations are available in Tables 2A, 3A and 4A in the Appendix. Here, in Table 4, we present only the main results, with $\gamma = 1$ for the FM-A and MBC VMP measures and with $\gamma = 0.8$ for the GM VMP measure as it was the γ value chosen in Gallardo (2020).

Table 4. Main VMP measures. Incidence rates and vulnerability to poverty ratios

VMP measurement	*Headcount ratio*	*Poverty-induced vulnerability*	*Risk-induced vulnerability*	*Vulnerability to poverty ratio*
FM	0.31	0.14	0.17	1.50
	(0.0027)	(0.0019)	(0.0023)	
FM-A (γ=1)	0.36	0.14	0.23	1.75
	(0.0025)	(0.0019)	(0.0024)	
GM (γ=0.8)	0.37	0.09	0.28	1.79
	(0.003)	(0.002)	(0.0048)	
MBC (γ=1)	0.33	0.17	0.16	1.59
	(0.0029)	(0.0023)	(0.0023)	

Note: Standard errors in parenthesis. For the FM measurement, poverty-induced vulnerability and risk-induced vulnerability correspond to high vulnerability and relative vulnerability. The poverty-induced vulnerability in the GM measurement was calculated using only the estimated probabilities of being non-deprived in each dimension, i.e., with $\gamma = 0$. The multidimensional poverty headcount ratio H is equal to 0.207.

The estimated VMP headcount ratio ranges between 0.31 in the FM measurement to 0.37 in the GM approach. Keeping in mind that the multidimensional poverty headcount ratio H is equal to 0.207, the VPR draws between 1.5 and 1.8 as is shown in the last column. Notably, all VMP headcount ratios presented in Table 4 are lower than those calculated by Gallardo (2020) using the same data but estimating the probabilities with Logit and Probit models. In that work, the VMP headcount ratios obtained were 0.39 for the GM-Probit model, 0.4 for the GM-Logit model and 0.43 for the FM-Probit model. Thus, the Bayesian classifiers estimator provides more conservative results.

Table 4 also offers the decompositions between poverty-induced and risk-induced vulnerability incidence rates. For the FM measurement, poverty-induced vulnerability and risk-induced vulnerability correspond to high vulnerability and relative vulnerability rates. In the GM measurement, poverty-induced vulnerability was calculated using only the estimated probabilities of being non-deprived in each dimension, i.e., with $\gamma = 0$, and then by residual the risk-induced vulnerability was obtained. It is noteworthy that the MBC measurement of vulnerability presents a poverty-induced vulnerability rate that is closest to the actual H value. This result is consistent with the better predictive performance of the Bayesian classifier Model 3. In contrast, for the other VMP measurements, the risk component of vulnerability appears bigger than the poverty component.

Both the GM and the MBC VMP measurements allow the estimation of the dimensional vulnerability rates presented in Table 5. These dimensional vulnerability indicators could be useful for policy makers to complement the information provided by the row deprivation incidence rate, also known as *uncensored headcount ratios*, in each dimension. We can see, for instance, that in some dimensions there are more vulnerable people than in others. The relation between poverty and vulnerability in each dimension could also be of policy interest. In this result for Chile, we observe that vulnerability rates in the dimensional indicators of child malnutrition, safety, employment and social security are quite high in comparison to their respective dimensional deprivation rates of effective poverty. In contrast, in the indicators of health insurance affiliation, access to healthcare and social participation and support, the vulnerability rates are close to their respective deprivation rates of poverty in these dimensions.

Likewise, the decomposition of dimensional vulnerability between their poverty and risk components could be of policy interest, as is shown in the last two columns of Table 5. In Chile, for instance, there are some dimensions in which this composition is more balanced (e.g., school attendance, schooling lag, years of adults' schooling, child malnutrition and health insurance affiliation); in other dimensions, this composition is more biased towards the risk component (e.g., child malnutrition, access to healthcare, employment, social security, retirement insurance and equal treatment). This information could be used to guide policy decisions. Thereby, in those dimensions for which vulnerability is more strongly determined by the poverty component, policies should be focussed on improving the average outcomes of the poor. On the other hand, in those dimensions for which vulnerability is more strongly influenced by the risk component, it is more important to implement and reinforce the insurance mechanisms.

Table 5. Dimensional vulnerability measures. Incidence rates and vulnerability to poverty ratios.

Dimension	UHR_m	$V^T_{GM,m}$	$V^T_{GM,m}/UHR_m$	$V^T_{MBC,m}$	$V^T_{MBC,m}/UHR_m$	$V^P_{MBC,m}$	$V^R_{MBC,m}$
School attendance	0.033	0.057	1.7	0.046	1.4	0.0248	0.0212
	(0.0014)	(0.0017)		(0.0015)		(0.0011)	(0.0011)
Schooling lag	0.030	0.053	1.7	0.047	1.6	0.0224	0.0245
	(0.0012)	(0.0015)		(0.0014)		(0.001)	(0.0011)
Years of schooling	0.319	0.484	1.5	0.485	1.5	0.2666	0.2180
	(0.0027)	(0.0028)		(0.0028)		(0.0025)	(0.0023)
Child malnutrition	0.068	0.150	2.2	0.150	2.2	0.0549	0.0950
	(0.0017)	(0.0022)		(0.0022)		(0.0015)	(0.0017)
Health insurance affiliation	0.061	0.062	1.0	0.068	1.1	0.0342	0.0343
	(0.0017)	(0.0019)		(0.002)		(0.0014)	(0.0014)
Access to healthcare	0.042	0.050	1.2	0.056	1.3	0.0190	0.0367
	(0.0012)	(0.0013)		(0.0014)		(0.0009)	(0.0011)
Employment	0.120	0.226	1.9	0.231	1.9	0.0718	0.1590
	(0.0021)	(0.0028)		(0.0028)		(0.0019)	(0.0024)
Social security	0.344	0.642	1.9	0.651	1.9	0.2342	0.4164
	(0.0029)	(0.003)		(0.0029)		(0.0025)	(0.003)
Retirement insurance	0.109	0.185	1.7	0.196	1.8	0.0618	0.1337
	(0.0018)	(0.0021)		(0.0022)		(0.0014)	(0.0018)
Habitability	0.213	0.388	1.8	0.352	1.7	0.1394	0.2124
	(0.0027)	(0.0029)		(0.0029)		(0.0023)	(0.0023)
Basic services	0.062	0.111	1.8	0.103	1.6	0.0589	0.0445
	(0.0011)	(0.0012)		(0.0011)		(0.0009)	(0.0008)
Local environment	0.099	0.143	1.4	0.143	1.4	0.0591	0.0837
	(0.0017)	(0.0021)		(0.0019)		(0.0014)	(0.0014)
Equal treatment	0.136	0.216	1.6	0.264	1.9	0.0739	0.1898
	(0.0026)	(0.0028)		(0.0029)		(0.0018)	(0.0027)
Safety	0.133	0.286	2.1	0.319	2.4	0.1225	0.1964
	(0.0021)	(0.0027)		(0.0027)		(0.0022)	(0.0025)
Social participation and support	0.064	0.057	0.9	0.074	1.1	0.0294	0.0447
	(0.0016)	(0.0016)		(0.002)		(0.0013)	(0.0016)

Note: Standard errors in parenthesis. UHR_m is the uncensored poverty headcount ratio in dimension m. $V^T_{GM,m}$ and $V^T_{MBC,m}$ are the incidence rates of total vulnerability according to the GM and MBC approaches, respectively. $V^P_{MBC,m}$ and $V^R_{MBC,m}$ are the poverty-induced and risk-induced vulnerability rates obtained for the MBC VMP decomposition. $V^T_{GM,m}$ and $V^T_{MBC,m}$ were estimated with $\gamma = 0.8$ and $\gamma = 1$, respectively.

In order to see how all these VMP measures are related to each other, we took advantage of the sub-group decomposition property that characterises the FGT poverty indices to calculate all these VMP measures for the 53 Chilean provinces. We then calculated how they correlate with each other and with the poverty headcount ratio H. These results joined with the means and the standard deviations of the VMP measures by province are presented in Table 6.

Table 6. Means, standard deviations and correlations with confidence intervals.

Variable	*Mean*	*Std. Dev.*	1	2	3	4
1. H	0.23	0.08				
2. FM-VMP	0.46	0.23	.60**			
			[.39, .75]			
3. FMA-VMP	0.52	0.24	.58**	1.00**		
			[.37, .74]	[.99, 1.00]		
4. GM-VMP	0.40	0.10	.63**	.78**	.78**	
			[.44, .77]	[.65, .87]	[.64, .87]	
5. MBC-VMP	0.36	0.09	.96**	.71**	.70**	.71**
			[.93, .98]	[.54, .82]	[.53, .82]	[.54, .82]

Note. H, FM-VMP, FMA-VMP, GM-VMP and *MBC-VMP SD* are used to represent the incidence rates of multidimensional poverty and vulnerability according to the corresponding measurement methods. The grey column indicates the correlations between vulnerability incidence rates and poverty incidence rate. Values in square brackets indicate the 95% confidence interval for each correlation, whereas * indicates $p < .05$. ** indicates $p < .01$.

Notably, the MBC VMP measure presents the highest correlation with the poverty headcount ratio, whereas both the MBC and the GM measures also exhibit less variability in comparison with the FM and FM-A measures. As can be expected between all estimated VMP measurements, a high correlation is observed (0.7 as minimum).

4. Concluding Remarks

Three Bayesian classifier strategies to estimate VMP have been introduced in this paper. According to the empirical example presented here, these machine-learning estimation strategies offer a good alternative to the traditional Probit and Logit models to predict the probability of being multidimensionally poor. Moreover, these are multidimensional strategies being used to solve a multidimensional estimation problem instead of facing a multidimensional problem by transforming it in one or several one-dimensional problems. The predictive performance of these Bayesian classifier models is high, and, therefore, they are suitable to be applied by policy-makers in practice. As we have shown, the Bayesian classifier models can be used to estimate both the FM and the GM VMP measurements

We have also introduced in this paper two new proposals for measuring vulnerability to multidimensional poverty, which include the downside mean-semideviation as the risk parameter. The first of these proposals is the FM-A, an asymmetric version of the FM measurement. In contrast with the previous FM measurement, this new proposal allows a clearer decomposition between poverty-induced and risk-induced vulnerability as this decomposition depends on a risk parameter and not on the current H value. Notably, as the vulnerability threshold of the FM-A does not depend on the H value as the FM does, it is more suitable for comparisons between time periods and geographical units. That is, the FM-A measurement is not data dependent.

The second VMP measurement procedure proposed in this paper is based on an MBC model that allows estimation of both the probability of being multidimensionally poor as well as the probability of being deprived in each welfare dimension simultaneously. This new approach offers some advantages in comparison with the other VMP measurements, first because it poses the highest predictive performance and the highest correlation with the effective multidimensional poverty itself and second because, in contrast with the GM measurement, it does not require the invocation of the AF counting method in the aggregation phase and instead is calculated directly from the probability of being multidimensionally poor/non-poor provided by the Bayesian network. Therefore, although this procedure can bring the richness of dimensional vulnerability estimates as the GM proposal does, at the same time, it also shares the parsimony of the FM approach.

Future research building on this paper should provide more empirical evidence of the performance of these VMP estimation proposals with massive data from several countries and regions and possibly use Bayesian networks to estimate the dynamic of multidimensional vulnerability across time.

References

Abraham, R., & Kavi, K. S. (2008). Multidimensional poverty and vulnerability. *Economic and Political Weekly*, 43(20), 79–87.

Asselin, L-M. (2009). *Analysis of multidimensional poverty*. Dordrecht: Springer.

Atkinson, A. B. (2003). Multidimensional deprivation. Contrasting social welfare and counting approaches. *Journal of Economic Inequality*, 1, 51–65.

Alkire, S., & Foster, J. (2011). Counting and multidimensional poverty measurement. *Journal of Public Economics*, 95 (7–8): 476–487.

Alkire, S., & Santos M. (2014). Measuring acute poverty in the developing world: Robustness and scope of the multidimensional poverty index. *World Development*, 59, 251–274.

Alkire, S., Foster, J., Seth, S., Santos, M., Roche, J., & Ballón, P. (2015). *Multidimensional poverty measurement and analysis*. Oxford: Oxford University Press.

Anand, S., & Sen, A. (1997). Concepts of human development and poverty: A multidimensional perspective. New York: UNDP.

Battiston, D., Cruces, G., Lopez-Calva, L. F., Lugo, M. A., & Santos, M. E. (2013). Income and beyond: Multidimensional poverty in six Latin American countries. *Social Indicators Research*, 112: 291–314.

Belhadj, B., & Limam, M. (2012). Unidimensional and multidimensional fuzzy poverty measures: New approach. Economic Modelling, 29, 995–1002.

Bielza, C., Li, G., & Larrañaga, P (2011). Multi-dimensional classification with Bayesian networks. *International Journal of Approximate Reasoning*, 52(6), 705–727.

Bielza, C., & Larrañaga, P. (2014). Discrete Bayesian network classifiers: A survey. *ACM Computing Surveys*, 47(1), 5.

Bossert, W., Chakravarty, S., & D'Ambrosio, C. (2013). Multidimensional poverty and material deprivation with discrete data. *Review of Income and Wealth*, 59(1), 29–43.

Bourguignon, F., & Chakravarty, S. (2003). The measurement of multidimensional poverty. *Journal of Economic Inequality*, 1(1), 25–49.

Calvo, C., & Dercon, S. (2005). Measuring individual vulnerability. University of Oxford, Department of Economics, Discussion Paper Series 229.

Calvo, C., & Dercon, S. (2007). Vulnerability to poverty. CSAE Working Paper 2007-03.

Calvo, C. (2008). Vulnerability to multidimensional poverty: Peru, 1998-2002. *World Development*, 36(6), 1011–1020.

Calvo, C., & Dercon, S. (2013). Vulnerability to individual and aggregate poverty. Social Choice and Welfare, 41, 721–740.

CASEN (2016). Metodología de medición de pobreza multidimensional con entorno y redes. Serie Documentos Metodológicos Casen, N°32.

Chakravarty, S. R., & Silber, J. (2008). Measuring multidimensional poverty: The axiomatic approach. In N. Kakwani & J. Silber (Eds.), *Quantitative approaches to multidimensional poverty measurement* (pp.192–209). New York: Palgrave Macmillan.

Chaudhuri, S., Jalan, J., & Suryahadi, A. (2002). Assessing household vulnerability to poverty from cross-sectional data: A methodology and estimates from

Indonesia. Department of Economics Discussion Paper Series (vol. 102), Columbia University.
Christiaensen, L., & Subbarao, K. (2005). Towards an understanding of household vulnerability in rural Kenya. Journal of African Economies, 14(4), 520–558.
Cowell, R., Dawid, A., Lauritzen, S., Spiegelhalter, D. (1999). *Probabilistic networks and expert systems*. New York: Springer.
Decancq, K., & Lugo, M.A. (2013). Weights in multidimensional indices of wellbeing: An overview. *Econometric Reviews*, 32(1), 7–34. https://doi.org/10.1080/07474938.2012.690641.
Doreswamy, D., & Hemanth K. (2011). Performance evaluation of predictive classifiers for knowledge discovery from engineering materials data sets. *Artificial Intelligent Systems and Machine Learning*, 3(3), 162–168.
Deutsch, J., & Silber, J. (2005). Measuring multidimensional poverty: An empirical comparison of various approaches. *Review of Income and Wealth*, 51(1), 145–174.
Dutta, I., Foster, J., & Mishra, A. (2011). On measuring vulnerability to poverty. *Social Choice and Welfare*, 37(4), 743–761.
Estrada, J. (2002). Systematic risk in emerging markets: The D-CAPM. *Emerging Markets Review*, 3, 365–379.
Feeny, S., & McDonald, L. (2016). Vulnerability to multidimensional poverty: Findings from households in Melanesia. *The Journal of Development Studies*, 52(3), 447–464.
Friedman, N., Geiger, D., & Goldszmidt, M. (1997). Bayesian network classifiers. *Machine Learning*, 29(2-3), 131–163.
Foster, J., Greer, J., & Thorbecke, E. (1984). A class of decomposable poverty measures. *Econometrica*, 52, 761–766.
Fauziyyah, N., Abdullah, S., & Nurrohmah, S. (2020). Reviewing the consistency of the naïve Bayes classifier's performance in medical diagnosis and prognosis problems. AIP Conference Proceedings. https://doi.org/10.1063/5.0007885.
Gallardo, M. (2013). Using the downside mean-semideviation for measuring vulnerability to poverty. *Economics Letters*, 120(3), 416–418.
Gallardo, M. (2018). Identifying vulnerability to poverty: A critical survey. *Journal of Economic Surveys*, 32(4): 1074-1105.
Gallardo, M. (2020). Measuring vulnerability to multidimensional poverty. *Social Indicators Research*, 148, 67–103.
Gil-Begue, S., Bielza, C., & Larrañaga, P. (2020). Multi-dimensional Bayesian network classifiers: A survey. *Articial Intelligence Review*, 54: 519–559.
Günther, I., & Harttgen, K. (2009). Estimating households' vulnerability to idiosyncratic and covariate shocks: A novel method applied in Madagascar. *World Development*, 37(7), 1222–1234.
Hohberg, M., Landau, K., Kneib, T., Klasen, S., & Zucchini, W. (2018).

Vulnerability to poverty revisited: Flexible modeling and better predictive performance. *Journal of Economic Inequality*, 16: 439–454.
Koller, D., & Friedman, N. (2009). *Probabilistic graphical models: Principles and techniques*. The MIT Press, Cambridge, Massachusetts, London, England.
Ogryczak, W., & Ruszczynski, A. (1999). From stochastic dominance to mean-risk models: Semideviations as risk measures. *European Journal of Operational Research*, 116, 33–50.
Ogryczak, W., & Ruszczynski, A. (2001). On consistency of stochastic dominance and mean-semideviations models. *Mathematical Programming*, 89, 217–232.
Pearl, J. (1988). *Probabilistic reasoning in intelligent systems: Networks of plausible inference*. Morgan Kaufmann Publishers, San Francisco, CA, United States.
Rish, I. (2001). An empirical study of the naive Bayes classifier. IJCAI 2001 Workshop on Empirical Methods in Artificial Intelligence.
Santos, M., & Villatoro, P. (2018). A multidimensional poverty index for Latin America. *Review of Income and Wealth*, 64(1): 52-82.
Santos, M. (2019). Challenges in designing national multidimensional poverty measures. Statistics Series, No. 100 (LC/TS.2019/5). Santiago: Economic Commission for Latin America and the Caribbean (ECLAC).
Suryahadi, A. & Sumarto, S. (2003). Poverty and vulnerability in Indonesia before and after the economic crisis. *Asian Economic Journal*, 17(1), 45–64.
Ting, S., Ip, W., & Tsang, A. (2011). Is naïve Bayes a good classifier for document classification? *International Journal of Software Engineering and Its Applications*, 5(3), 37-46.
Van der Gaag, L., & De Waal, P. (2006). Multi-dimensional Bayesian network classifiers. Third European Conference on Probabilistic Graphical Models, 107–114.
World Bank (2020). *Poverty and shared prosperity 2020: Reversals of fortune*. Washington, DC: World Bank. doi: 10.1596/978-1-4648-1602-4.

Appendix

Table A1. Deprivation cut-off of the Chilean MPI

Indicator	Deprivation cut-off
School attendance	At least one child aged 4-18 years is not attending school and has not completed middle school, or at least one person from 6 to 26 years with permanent disability does not attend school.
Schooling lag	At least one person under 22 years of age is still attending school and is two or more years behind.
Years of schooling	At least one person over the age of 18 has attained fewer years of schooling than those established by law.
Child malnutrition	At least one child aged 0-6 years is overweight or obese or is malnourished or at risk of malnutrition.
Health insurance affiliation	At least one person does not have health insurance.
Access to healthcare	At least one person for reasons beyond their control did not have access to healthcare in the last 3 months or did not have coverage under the AUGE-GES system for the treatment of chronic diseases.
Occupation	At least one person over 18 is unemployed.
Social security	A person of 15 years or more who is employed is not registered in saving for the pension system and is not an independent worker with full higher education.
Retirement	A person of retirement age does not receive a pension and does not receive other income from leases, profits, dividends or interest.
Habitability	There is a situation of overcrowding (number of people per bedroom is greater than or equal to 2.5) or residence in a precarious house.
Basic services	Residence in a house without basic sanitary services (WC, key inside the house and water according to urban or rural standards).
Local environment	There are 2 or more environmental pollution problems that frequently occur in the area of residence or there is a lack in the area of residence of some basic equipment (health, education and transport); either all in the household are unemployed or, being employed, take an average of 1 hour or more in public transport to move from their home to their work.
Social participation and support	The existence of no person that can help outside the members of the household in 8 relevant situations of support or care; no household members 14 years or older who have participated in any social organisation in the last 12 months.

Equal treatment	Someone in the household has received discriminatory or unfair treatment during the last 12 months.
Safety	Someone in the household has witnessed 'always' drug trafficking or shootings during the last month.

Table 2A. Gallardo VMP measures for different risk aversion parameters γ .

γ -value	VH	VP	VR
0.1	0.102	0.093	0.009
	(0.002)	(0.002)	
0.2	0.114	0.093	0.021
	(0.0021)	(0.002)	
0.3	0.131	0.093	0.038
	(0.0022)	(0.002)	
0.4	0.156	0.093	0.063
	(0.0023)	(0.002)	
0.5	0.190	0.093	0.097
	(0.0024)	(0.002)	
0.6	0.237	0.093	0.144
	(0.0026)	(0.002)	
0.7	0.301	0.093	0.208
	(0.0028)	(0.002)	
0.8	0.371	0.093	0.278
	(0.003)	(0.002)	
0.9	0.448	0.093	0.354
	(0.003)	(0.002)	
1	0.521	0.093	0.428
	(0.003)	(0.002)	

Table 3A. FM-A VMP measures for different risk aversion parameters γ .

γ -value	V0	VP	VR	V1	V2
0.1	0.151	0.136	0.015	0.066	0.040
	(0.0018)	'(0.0017)	(0.0007)	(0.0009)	(0.0006)
0.2	0.168	0.136	0.032	0.074	0.045
	(0.0019)	'(0.0017)	(0.001)	(0.0009)	(0.0007)
0.3	0.186	0.136	0.050	0.083	0.051
	(0.002)	'(0.0017)	(0.0013)	(0.001)	(0.0007)
0.4	0.207	0.136	0.071	0.094	0.059
	(0.0021)	'(0.0017)	(0.0015)	(0.001)	(0.0008)
0.5	0.230	0.136	0.094	0.106	0.068
	(0.0022)	'(0.0017)	(0.0017)	(0.0011)	(0.0008)
0.6	0.255	0.136	0.119	0.120	0.078
	(0.0023)	'(0.0017)	(0.0019)	(0.0012)	(0.0009)
0.7	0.281	0.136	0.145	0.136	0.091
	(0.0024)	'(0.0017)	(0.002)	(0.0013)	(0.001)
0.8	0.307	0.136	0.171	0.154	0.106
	(0.0024)	'(0.0017)	(0.0022)	(0.0014)	(0.0011)
0.9	0.335	0.136	0.199	0.174	0.123
	(0.0025)	'(0.0017)	(0.0023)	(0.0015)	(0.0012)
1	0.362	0.136	0.226	0.196	0.143
	(0.0025)	'(0.0017)	(0.0024)	(0.0016)	(0.0013)

Table 4A. MBC VMP measures for different risk aversion parameters γ .

γ -value	V0	VP	VR	V1	V2
0.1	0.176	0.166	0.011	0.098	0.071
	(0.0024)	(0.0023)	(0.0006)	(0.0016)	(0.0014)
0.2	0.188	0.166	0.022	0.105	0.076
	(0.0024)	(0.0023)	(0.0008)	(0.0016)	(0.0014)
0.3	0.197	0.166	0.032	0.113	0.082
	(0.0025)	(0.0023)	(0.001)	(0.0017)	(0.0015)
0.4	0.208	0.166	0.043	0.122	0.090
	(0.0025)	(0.0023)	(0.0012)	(0.0017)	(0.0015)
0.5	0.220	0.166	0.054	0.132	0.098
	(0.0025)	(0.0023)	(0.0013)	(0.0018)	(0.0016)
0.6	0.233	0.166	0.068	0.142	0.108
	(0.0026)	(0.0023)	(0.0015)	(0.0019)	(0.0016)
0.7	0.254	0.166	0.088	0.154	0.119
	(0.0027)	(0.0023)	(0.0017)	(0.0019)	(0.0017)
0.8	0.273	0.166	0.107	0.168	0.131
	(0.0028)	(0.0023)	(0.002)	(0.002)	(0.0018)
0.9	0.299	0.166	0.133	0.183	0.146
	(0.0028)	(0.0023)	(0.0021)	(0.0021)	(0.0019)
1	0.329	0.166	0.163	0.200	0.162
	(0.0029)	(0.0023)	(0.0023)	(0.0022)	(0.002)