Obvious manipulations of tops-only voting rules*

R. Pablo Arribillaga† Agustín G. Bonifacio†

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Abstract

In a classical voting problem with a finite set of (at least three) alternatives to choose from, we study the manipulation of tops-only and unanimous rules. Since strategy-proofness is impossible to obtain on the universal domain of (strict) preferences, we investigate the weaker concept of non-obvious manipulability (NOM). First, we show that NOM is equivalent to every veto from any agent being a strong veto. Second, we focus on two classes of tops-only rules: (i) (generalized) median voter schemes, and (ii) voting by committees. For each class, we identify which rules satisfy NOM on the universal domain of preferences.

JEL classification: D71.

Keywords: obvious manipulations, tops-onlyness, unanimity, median voting schemes, voting by committees.

1 Introduction

Voting rules are procedures that allow a group of agents to select an alternative, among many, according to their preferences. Within desirable properties a voting rule may satisfy, the concept of strategy-proofness has played a vital role for studiyng the strategic

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†Instituto de Matemática Aplicada San Luis, Universidad Nacional de San Luis and CONICET, San Luis, Argentina, and RedNIE. Emails: rarribi@unsl.edu.ar and abonifacio@unsl.edu.ar

behavior of the agents. A voting rule is strategy-proof if it is always in the best interest of the agents to reveal their true preferences. Unfortunately, outside of a dictatorship, there is no strategy-proof voting rule when more than two alternatives, and all possible preferences over alternatives, are considered (Gibbard, 1973; Satterthwaite, 1975).

Two main approaches have been taken to circumvent Gibbard-Satterthwaite's impossibility theorem. The first approach restricts the domain of preferences that agents can have over alternatives (see Barberà, 2011, and references therein). The second approach considers weakenings of strategy-proofness, and has been an active field of research in recent years.

Troyan and Morrill (2020) introduce the concept of obvious manipulation in the context of market design. They assume that an agent knows the possible outcomes of the mechanism conditional on his own declaration of preferences, and define a deviation from the truth to be an obvious manipulation if either the best possible outcome under the deviation is strictly better than the best possible outcome under truth-telling, or the worst possible outcome under the deviation is strictly better than the worst possible outcome under truth-telling. A mechanism that does not allow any obvious manipulation is called not obviously manipulable.

In this paper we study (not) obvious manipulation of voting rules when a finite set of alternatives is involved. We focus on tops-only rules: rules that only consider agents' top alternatives in order to select a social choice.

Our main result gives a characterization of not obviously manipulable rules in terms of veto power of the agents. An agent vetoes an alternative if there is a preference report of the agent that forces the rule to never select such alternative. The veto is strong if the report of *any* preference with top different from the alternative forces the rule to never select it. Theorem 1 states that, within tops-only and unanimous[1] rules, not obvious manipulation is equivalent to each veto being a strong veto.

Next, we apply our main result to study well-known classes of rules defined on the universal domain of preferences: (generalized) median voter schemes and voting by committees. On the restricted domain of single-peaked preferences, median voter schemes are the only tops-only, unanimous, and anonymous rules that are also strategy-proof. Generalized median voter schemes are the only ones that satisfy all such properties except anonymity. On the domain of separable preferences, voting by committees are the only

[1]If an alternative is most preferred by all agents, the rule should select it.

tops-only and unanimous rules that are also strategy-proof.

Note, however, that if the designer cannot guarantee that the domain restrictions are met and the full domain of preferences has to be considered, then strategy-proofness no longer holds. For this reason, it is important to identify which rules within these families obey the less demanding property of non-obvious manipulability.

In Theorem 2, we show that if $X = \{a, a+1, \ldots, b\}$ is the set of alternatives and α_1 and α_{n-1}, with $\alpha_1 \leq \alpha_{n-1}$, are the extremal fixed ballots associated to median voter scheme f, then f is not obviously manipulable if and only if $\alpha_1 \in \{a, a+1\}$ and $\alpha_{n-1} = b$ or $\alpha_1 = a$ and $\alpha_{n-1} \in \{b-1, b\}$. A similar condition applied to the extremal fixed ballots (for each agent) in the monotonic family of fixed ballots associated to generalized median voter schemes characterizes not obviously manipulable ones (Theorem 3).

In Theorem 4, we show that a voting by committees is non-obvious manipulable if and only if no agent is a vetoer. In terms of the committees defining the rule, this is equivalent to say that: (i) no agent belongs to all committees, and (ii) minimal committees have at least two members. When anonymity is added to the picture, voting by committees simplify to voting by quota. In this case, we prove that non-obvious manipulability is equivalent to each committee having quota between 2 and $n-1$ (Corollary 3).

The paper of Aziz and Lam (2021) is the closest to ours and, to the best of our knowledge, is the first one that applies Troyan and Morrill (2020) notion in the context of voting. Aziz and Lam (2021) present a general sufficient condition for a voting rule to be not obviously manipulable. However, they focus on non-tops-only rules. They show that Condorcet consistent as well as some other strict scoring rules are not-obviously-manipulable. Furthermore, for the class of k-approval voting rules, they give necessary and sufficient conditions for obvious manipulability. Other recent papers that study obvious manipulations, in contexts other than voting, are Ortega and Segal-Halevi (2022) and Psomas and Verma (2022).

The rest of the paper is organized as follows. The model and the concept of obvious manipulations are introduced in Section 2. In Section 3, we present the main result of our paper that characterizes non-obvious manipulable rules. Section 4 deals with applications: in Subsection 4.1 we study (generalized) median voter schemes, and in Subsection 4.2 we study voting by committees. To conclude, some final remarks are gathered in Section 5.

2 Preliminaries

2.1 Model

A set of *agents* $N = \{1, \ldots, n\}$, with $n \geq 2$, has to choose an alternative from a finite and given set X (with cardinality $|X| = m \geq 2$). Each agent $i \in N$ has a strict *preference* P_i over X. Denote by $t(P_i)$ to the best alternative according to P_i, called the *top* of P_i; and by $b(P_i)$ to the worst alternative according to P_i, called the *bottom* of P_i. We denote by R_i the weak preference over X associated to P_i; *i.e.,* for all $x, y \in X$, xR_iy if and only if either $x = y$ or xP_iy. Let $\mathcal{P}$ be the set of all strict preferences over X. A (preference) *profile* is a n-tuple $P = (P_1, \ldots, P_n) \in \mathcal{P}^n$, an ordered list of n preferences, one for each agent. Given a profile P and an agent i, P_{-i} denotes the subprofile in $\mathcal{P}^{n-1}$ obtained by deleting P_i from P.

A *(social choice) rule* is a function $f : \mathcal{P}^n \longrightarrow X$ selecting an alternative for each preference profile in $\mathcal{P}^n$. The range of a rule $f : \mathcal{P}^n \longrightarrow X$ is denoted by r_f.

A rule $f : \mathcal{P}^n \longrightarrow X$ is *unanimous* if for all $P \in \mathcal{P}^n$ such that $t(P_i) = x$ for all $i \in N$, $f(P) = x$. A rule $f : \mathcal{P}^n \longrightarrow X$ is *efficient* if for all $R \in \mathcal{P}^n$, there is no $x \in X$ such that, for all $i \in N$, $xP_if(R)$ and $xP_jf(P)$ for some $j \in N$ (note that efficiency implies unanimity). A rule $f : \mathcal{P}^n \longrightarrow X$ is *tops-only* if for all $P, P' \in \mathcal{P}^n$ such that $t(P_i) = t(P_i')$ for all $i \in N$, $f(P) = f(P')$. In this paper we will focus in unanimous and tops-only[2] social choice function and in the rest of the paper we always assume that both properties are satisfied for any rule.

Rules require each agent to report a preference on a domain $\mathcal{P}$. Given $P_i \in \mathcal{P}$ an alternative report P_i' is a *(profitable) manipulation of rule f at P_i* if there is a preference subprofile $P_{-i} \in \mathcal{P}^{n-1}$ such that

$$f(P_i', P_{-i})P_if(P_i, P_{-i}).$$

A rule is strategy-proof on $\mathcal{P}^n$ if no agent has a manipulation. Formally, a rule $f : \mathcal{P}^n \longrightarrow X$ is *strategy-proof* if for all $P \in \mathcal{P}^n$, all $i \in$ N, and all $P_i' \in \mathcal{P}$,

$$f(P_i, P_{-i})R_if(P_i', P_{-i}).$$

Other desirable properties we look at are the following. A rule $f : \mathcal{P}^n \longrightarrow X$ is *anonymous* if it is invariant with respect to the agents' names; namely, for all one-to-one mappings $\sigma : N \longrightarrow N$ and all $P \in \mathcal{P}^n$, $f(P_1, \ldots, P_n) = f(P_{\sigma(1)}, \ldots, P_{\sigma(n)})$. A rule $f : \mathcal{P}^n \longrightarrow X$ is *dictatorial* if there exists $i \in N$ such that for all $R \in \mathcal{P}^n$, $f(P)R_ix$ for all $x \in r_f$.

[2]For a treatment of non tops-only rules see Aziz and Lam (2021)

The Gibbard-Satterthwaite Theorem states that a rule $f : \mathcal{P}^n \longrightarrow X$, with $|r_f| > 2$, is strategy-proof if and only if it is dictatorial (Gibbard, 1973; Satterthwaite, 1975). This negative result justifies the study of less demanding criteria of (lack of) manipulation when rules defined on the universal domain of preferences are considered. One such weakening of strategy-proofness is presented next.

2.2 Obvious manipulations

The notion of obvious manipulations has been introduced by Troyan and Morrill (2020) in the context of matching model and then by Aziz and Lam (2021) in the context of voting. In order to introduce the notion of obvious manipulations, we consider the set of alternatives that the rule could choose once that agent i fix its preference. Formally, given a preference $P_i \in \mathcal{P}$, we define the *option set* of f at P_i as

$$O^f(P_i) = \{f(P_i, P_{-i}) \in X : P_{-i} \in \mathcal{P}^{n-1}\}.$$

Now, a manipulation is obvious if the pessimistic or optimistic scenario under the manipulation provides the agents with a better result than the one he would obtain under truth-telling.

Definition 1 *(Troyan and Morrill, 2020) Let $f : \mathcal{P}^n \longrightarrow X$ be a rule, let $P_i \in \mathcal{P}$, and let $P_i' \in \mathcal{P}$ be a profitable manipulation of f at P_i. Manipulation P_i' is **obvious** if*

$$\min_{P_i} O^f(P_i') \; P_i \; \min_{P_i} O^f(P_i). \tag{1}$$

or

$$\max_{P_i} O^f(P_i') \; P_i \; \max_{P_i} O^f(P_i). \tag{2}$$

*Rule f is **not obviously manipulable (NOM)** if it does not admit any obvious manipulation.*

Remark 1 *Condition (2) is irrelevant when the rule satisfies unanimity because $\max_{P_i} O^f(P_i) = \{t(P_i)\}$. So, under unanimity, a profitable manipulation P_i' of rule f at P_i is obvious if and only if (1) holds. These manipulations are called worst case obvious manipulation in Aziz and Lam (2021).*

3 Main theorem

In order to obtain our main result, we first need to define when an agent has veto power. An agent vetoes an alternative if there is a preference report of the agent that forces the rule to never select such alternative. The veto is strong if the report of *any* preference with top different from the alternative forces the rule to never select it. Formally,

Definition 2 *Let $f : \mathcal{P}^n \longrightarrow X$ be a rule and let $i \in N$, $x \in X$, and $P_i \in \mathcal{P}$. Agent i **vetoes** x **via** P_i if $x \notin O^f(P_i)$. Agent i **strongly vetoes** x if i vetoes x via any preference whose top is different from x.*

Denote by V_i the set of all alternatives that agent i vetoes via some preference and by SV_i the set of all alternatives strongly vetoed by agent i. Note that $SV_i \subseteq V_i$. Given $x \in V_i$, let $\mathcal{V}_i^x = \{P_i \in \mathcal{P} : i \text{ vetoes } x \text{ via } P_i\}$ be the set of all preferences by which x is vetoed by agent i. Clearly the sets V_i, SV_i and $\mathcal{V}_i^x$ depend on f but we omit this reference to ease notation.

Theorem 1 *A tops-only and unanimous rule is NOM if and only if every veto is a strong veto, i.e., $V_i = SV_i$ for each agent $i \in N$.*

Proof. Let $f : \mathcal{P}^n \longrightarrow X$ be a tops-only and unanimous rule.
($\Longrightarrow$) Assume there is $i \in N$ such that $V_i \neq SV_i$. Since $SV_i \subseteq V_i$, $V_i \neq \emptyset$ and there is $x \in X$ such that $x \in V_i \setminus SV_i$. First, notice that

$$\mathcal{V}_i^x \subsetneq \{P_i \in \mathcal{P} : t(P_i) \neq x\}. \tag{3}$$

To see this, assume $P_i \in \mathcal{V}_i^x$ is such that $t(P_i) = x$ and let $P_{-i} \in \mathcal{P}^{n-1}$ be such that $t(P_j) = x$ for each $j \in N \setminus \{i\}$. Then, by unanimity, $f(P) = x$ so $x \in O^f(P_i)$ and $x \notin V_i$, a contradiction. Thus, $\mathcal{V}_i^x \subsetneq \{P_i \in \mathcal{P} : t(P_i) \neq x\}$. Moreover, since $x \in V_i \setminus SV_i$, (3) holds. Therefore, there is $P_i \in \mathcal{P}$ such that $t(P_i) \neq x$ and $P_i \notin \mathcal{V}_i^x$. Thus, there exists $P_{-i} \in \mathcal{P}^{n-1}$ such that $f(P_i, P_{-i}) = x$. Next, let $\overline{P}_i \in \mathcal{P}$ be such that $t(\overline{P}_i) = t(P_i)$ and $b(\overline{P}_i) = x$. By tops-onlyness, $f(\overline{P}_i, P_{-i}) = x$. Since $x \in V_i$, there is $P_i' \in \mathcal{P}$ such that $x \notin O^f(P_i')$. Then, $f(P_i', P_{-i}) \neq x$ and therefore $f(P_i', P_{-i})\overline{P}_i x = f(\overline{P}_i, P_{-i})$, implying that P_i' is a profitable manipulation of f at P_i. Furthermore, as $x \notin O^f(P_i')$,

$$\min_{\overline{P}_i} O^f(P_i')\ \overline{P}_i\ x = \min_{\overline{P}_i} O^f(\overline{P}_i).$$

Thus, P_i' is an obvious manipulation of f.

($\Longleftarrow$) Let $i \in N$ be such that $V_i = SV_i$. We will prove that i does not have an obvious manipulation. If $V_i = \emptyset$, the proof is trivial. Assume that $V_i \neq \emptyset$. Let $P_i \in \mathcal{P}$ be an arbitrary preference for agent i. There are two cases to consider:

1. $\boldsymbol{t(P_i) \notin V_i}$. Then, $t(P_i) \neq x$ for each $x \in V_i$. As i strongly vetoes any $x \in V_i$, $P_i \in \mathcal{V}_i^x$ for each $x \in V_i$. Therefore, $O^f(P_i) = X \setminus V_i$. Furthermore, by definition of V_i, $X \setminus V_i \subseteq O^f(P_i')$ for each $P_i' \in \mathcal{P}$. Thus, $\min_{P_i} O^f(P_i)\ R_i\ \min_{P_i} O^f(P_i')$ for each $P_i' \in \mathcal{P}$.

2. $\boldsymbol{t(P_i) \in V_i}$. As $V_i = SV_i$, $P_i \in \mathcal{V}_i^x$ for each $x \in V_i \setminus \{t(P_i)\}$. Therefore, by unanimity, $O^f(P_i) = (X \setminus V_i) \cup \{t(P_i)\}$. Furthermore, by definition of V_i, $X \setminus V_i \subseteq O^f(P_i')$ for each $P_i' \in \mathcal{P}$. Then, $\min_{P_i} O^f(P_i) = \min_{P_i}(X \setminus V_i)\ R_i\ \min_{P_i} O^f(P_i')$ for each $P_i' \in \mathcal{P}$.

Hence, i does not have an obvious manipulation. □

Corollary 1 *An efficient and tops-only rule is NOM if and only if some of the following statements hold:*

(i) There is at most one $i \in N$ such that $V_i \neq \emptyset$ and, moreover, $V_i = SV_i$.

(ii) There is $y \in X$ such that $SV_i = V_i \subseteq \{y\}$, for each $i \in N$.

Proof. Let $f : \mathcal{P}^n X$ be an efficient and tops-only rule.
($\Longrightarrow$) Assume both conditions (i) and (ii) do not hold. Then, there are distinct $i, j \in N$ and distinct $x, y \in X$ such that $x \in V_i$ and $y \in V_j$. Now let $P \in \mathcal{P}^n$ be such that $P_i : y, x, \dots$ and $P_k : x, y, \dots$ for each $k \in N \setminus \{i\}$. By efficiency, $f(P) \in \{x, y\}$. Therefore, $P_i \notin \mathcal{V}_i^x$ or $P_j \notin \mathcal{V}_i^y$. So, by Theorem 1, f is not NOM.
($\Longleftarrow$) By Theorem 1 it is clear that either condition is sufficient for f to be NOM. □

By Corollary 1 we have that under efficiency and tops-onlyness NOM implies a very limited veto power; at most one agent could veto some alternatives or only one alternative can be vetoed by some agents.

Corollary 2 *An efficient, anonymous and tops-only rule is NOM if and only if either $V_i = \emptyset$ for each $i \in N$ or there is $y \in X$ such that $SV_i = V_i = \{y\}$ for each $i \in N$.*

Proof. It follows from Corollary 1 and anonymity. □

Corollary 2 states that, under efficiency and unanimity, non-obvious manipulability is equivalent to having at most one alternative vetoed and that, if there is one such alternative, the veto is unanimous.

4 Applications

In this section, we apply Theorem 1 to study two classes of tops-only and unanimous voting rules in two separate (but related) voting problems. Our results allow us to discriminate those rules in each class which are non-obviously manipulable in the universal domain of preferences.

In the first problem, presented in subsection 4.1, alternatives are endowed with a linear order structure. When preferences are single peaked over that order, the family of (generalized) voting schemes encompass all strategy-proof rules. In the second problem, presented in subsection 4.2, alternatives consist of *subsets* of objects chosen from a fixed finite set. When preferences are separable, the class of voting by committees encompass all strategy-proof rules.

4.1 Median Voter Schemes

In this subsection assume that $X = \{a, a+1, a+2, \ldots, b\} \subset \mathbb{Z}$ and $b = a + (m-1)$. A preference $P_i \in \mathcal{P}$ is *single-peaked* on X if for all $x, y \in X$ such that $x \leq y < t(P_i)$ or $t(P_i) < y \leq x$, we have $t(P_i) P_i y P_i x$. We denote the domain of all single-peaked preferences on X by $\mathcal{SP}$. Note that $\mathcal{SP} \subsetneq \mathcal{P}$.

Moulin (1980) characterizes the family of strategy-proof, unanimous, and tops-only rules on the domain of single-peaked preferences. This family contains many non-dictatorial rules. All of them are extensions of the median voters. Following Moulin (1980), and before presenting the general result, we first introduce the anonymous subclass characterizing those rules which are NOM. After that, we present the general class of all strategy-proof, unanimous and tops-only rules on $\mathcal{SP}^n$ and characterize those which are NOM when they operate on the domain $\mathcal{P}^n$.

4.1.1 Anonymity

Assume first that n is odd and let $f : \mathcal{P}^n \longrightarrow X$ be the rule that selects, for each preference profile $P = (P_1, \ldots, P_n) \in \mathcal{P}^n$, the median among the top alternatives of the n agents; namely, $f(P) = med\{t(P_1), \ldots, t(P_n)\}$.[3] This social choice function is anonymous, effi-

[3]Given a set of real numbers $\{x_1, \ldots, x_K\}$, where K is odd, define its *median* as $med\{x_1, \ldots, x_K\} = y$, where y is such that $|\{1 \leq k \leq K : x_k \leq y\}| \geq \frac{K}{2}$ and $|\{1 \leq k \leq K : x_k \geq y\}| \geq \frac{K}{2}$. Since K is odd the median

cient, tops-only, and strategy-proof on $\mathcal{SP}$. Add now, to the n agents' top alternatives, $n-1$ fixed ballots: $\frac{n-1}{2}$ ballots at alternative a and $\frac{n-1}{2}$ ballots at alternative b. Then, the median among the n top alternatives, and the median among the n top alternatives and the $n-1$ fixed ballots coincide since the $\frac{n-1}{2}$ ballots at a and the $\frac{n-1}{2}$ ballots at b cancel each other; namely, for all $P = (P_1, \ldots, P_n) \in \mathcal{P}^n$,

$$f(P) = med\{t(P_1), \ldots, t(P_n), \underbrace{a, \ldots\ldots, a}_{\frac{n-1}{2}-\text{times}}, \underbrace{b, \ldots\ldots, b}_{\frac{n-1}{2}\text{-times}}\} = med\{t(P_1), \ldots, t(P_n)\}.$$

To proceed, and instead of adding $n-1$ fixed ballots at the extremes of the interval, we can add, regardless of whether n is odd or even, $n-1$ fixed ballots at any of the alternatives in X. Then, a rule $f : \mathcal{P}^n \longrightarrow X$ is a *median voter scheme* if there exist $n-1$ fixed ballots $(\alpha_1, \ldots, \alpha_{n-1}) \in X^{n-1}$ such that for all $P \in \mathcal{P}^n$,

$$f(P) = med\{t(P_1), \ldots, t(P_n), \alpha_1, \ldots, \alpha_{n-1}\}. \tag{4}$$

Proposition 1 *(Moulin, 1980) Rule $f : \mathcal{SP}^n \longrightarrow X$ is strategy-proof, tops-only, unanimous, and anonymous if and only if it is a median voter scheme.*

Given a vector $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in X^{n-1}$ of fixed ballots (w.l.o.g we assume that $\alpha_1 \leq \ldots \leq \alpha_{n-1}$) we denote by f^α its associated median voter scheme on $\mathcal{P}^n$; namely, for all $P \in \mathcal{P}^n$,

$$f^\alpha(P) = med\{t(P_1), \ldots, t(P_n), \alpha_1, \ldots, \alpha_{n-1}\}.$$

Furthermore, to simplify notation we use $O^\alpha(P_i)$ instead of $O^{f^\alpha}(P_i)$.

Remark 2 *By definition of option sets and f^α, it is easy to see that $\{x \in X : \alpha_1 \leq x \leq \alpha_{n-1}\} \subseteq O^\alpha(P_i)$ for each $P_i \in \mathcal{P}$.*

Median voter schemes are strategy-proof on the domain $\mathcal{SP}^n$ of single-peaked preferences. However, when they operate on the larger domain $\mathcal{P}^n$ they may become manipulable. Then, all median voter schemes are equivalent from the classical manipulability point of view. In this subsection we give a simple test to identified which median voter schemes are NOM.

Lemma 1 *Let $f^\alpha : \mathcal{P}^n \longrightarrow X$ be a median voter scheme and let $i \in N$. Then, $x \in V_i$ if and only if either $x < \alpha_1$ or $x > \alpha_{n-1}$.*

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is unique and belongs to the set $\{x_1, \ldots, x_K\}$.

Proof. ($\Longrightarrow$) Assume that $x \in V_i$. Then, there is $P_i \in \mathcal{P}$ such that $x \notin O^{\alpha}(P_i)$. Thus, by Remark 2, either $x < \alpha_1$ or $x > \alpha_{n-1}$.

($\Longleftarrow$) First, assume $x < \alpha_1$ and let $P_i \in \mathcal{P}$ be such that $t(P_i) = \alpha_1$. Then, $x \notin \{\alpha_1, \alpha_1 + 1, \ldots, b\} = O^{\alpha}(P_i)$. Thus, $x \in V_i$. Now, let $x > \alpha_{n-1}$ and let $P_i \in \mathcal{P}$ be such that $t(P_i) = \alpha_{n-1}$. Then, $x \notin \{a, a+1, \ldots, \alpha_{n-1}\} = O(P_i)$. Thus, $x \in V_i$. □

Theorem 2 *A median voter scheme $f^{\alpha} : \mathcal{P}^n \longrightarrow X$ is NOM if and only if*

(i) $\alpha_1 \in \{a, a+1\}$ *and* $\alpha_{n-1} = b$, *or*

(ii) $\alpha_1 = a$ *and* $\alpha_{n-1} \in \{b-1, b\}$.

Proof. **Claim**: Assume that $x < \alpha_1$ ($x > \alpha_{n-1}$). Then agent i strongly vetoes x if and only if $x = a$ and $\alpha_{n-1} = b$ ($x = b$ and $\alpha_1 = a$).

Assume that $x < \alpha_1$ (the proof for $x > \alpha_{n-1}$ is simetric and therefore it is ommited). Now assume that $a < x < \alpha_1$. Let P_i such that $t(P_i) = a$, then $O^{\alpha}(P_i) = \{a, a+1, \ldots, \alpha_{n-1}\}$ and $x \in O^{\alpha}(P_i)$. Thus, i does not strongly vetoes x. Now assume that $x = a$ and $\alpha_{n-1} < b$. Let P_i such that $t(P_i) = b$, then $O^{\alpha}(P_i) = \{\alpha_1, \alpha_1 + 1, \ldots, b\}$ and $x \in O^{\alpha}(P_i)$. Thus, i does not strongly vetoes x. Therefore, if i strongly vetoes x we have that $x = a$ and $\alpha_{n-1} = b$.

Now supose that $a = x < \alpha_1$ and $\alpha_{n-1} = b$. Give any profile $P_i \in \mathcal{P}$ such that $t(P_i) \neq x$ we have $O^{\alpha}(P_i) = \{\min\{t(P_i), \alpha_1\}, \}$. Thus, $x \notin O^{\alpha}(P_i)$. Hence, i strongly vetoes x. It finishes the proof of the Claim.

Now the proof of Collorary follows from Theorem 1, Lemma 1 and the Claim. □

4.1.2 General Case

Now we present the characterization of all strategy-proof, tops-only and unanimous rules on the domain of single-peaked preferences for all $n \geq 2$. We say that a collection $p = \{p_S\}_{S \in 2^N}$ is a *monotonic family of fixed ballots* if (i) $p_S \in X$ for all $S \in 2^N$ with $p_N = a$ and $p_{\varnothing} = b$ and (ii) $T \subseteq Q$ implies $p_Q \leq p_T$. A rule $f : \mathcal{P}^n \longrightarrow X$ is a *generalized median voter scheme* if there exits a monotonic family of fixed ballots $p = \{p_S\}_{S \in 2^N}$ such that, for each $P \in \mathcal{P}^n$,

$$f(P) = \min_{S \in 2^N} \max_{j \in S} \{t(P_j), p_S\}.$$

Proposition 2 (Moulin, 1980) *Rule $f : \mathcal{SP}^n \to X$ is strategy-proof, tops-only, and unanimous if and only if it is a generalized median voter scheme.*

Given a monotonic family of fixed ballots $p = \{p_S\}_{S \in 2^N}$ let f^p be its associated generalized median voter scheme. To simplify notation we use $O^p(P_i)$ instead of $O^{f^p}(P_i)$.

Remark 3 *By monotonicity of p, $p_{N\setminus\{i\}} \leq p_S$ for each S such that $i \notin S$ and $p_T \leq p_{\{i\}}$ for each T such that $i \in T$. Assume that $p_{N\setminus\{i\}} \leq p_{\{i\}}$. Let $x \in X$ such that $p_{N\setminus\{i\}} \leq x \leq p_{\{i\}}$ and let $P_{-i} \in \mathcal{P}^{n-1}$ be such that $t(P_j) = x$ for each $j \in N \setminus \{i\}$. Then, $f^p(P_i, P_{-i}) = x$ for all $P_i \in \mathcal{P}$. Therefore, $\{x \in X : p_{N\setminus\{i\}} \leq x \leq p_{\{j\}}\} \subseteq O^p(P_i)$ for each $P_i \in \mathcal{P}_i$.*

Lemma 2 *Let $f^p : \mathcal{P}^n \longrightarrow X$ be a generalized median voter scheme and let $i \in N$.*

(i) If $p_{N\setminus\{i\}} \leq p_{\{i\}}$, then $x \in V_i$ if and only if either $x < p_{N\setminus\{i\}}$ or $x > p_{\{i\}}$.

(ii) If $p_{\{i\}} < p_{N\setminus\{i\}}$, then $V_i = X$.

Proof. Let $f^p : \mathcal{P}^n \longrightarrow X$ be a generalized median voter scheme and let $i \in N$.
(i) The proof follows a similar argument to that used in the proof of Lemma 1 (invoking Remark 3 instead of Remark 2 and with $p_{N\setminus\{i\}}$ playing the role of α_1 and $p_{\{i\}}$ playing the role of α_{n-1}).
(ii) Let $x \in X$. There are two cases to consider:

1. $\boldsymbol{p_{\{i\}} < x}$. Let $P_i \in \mathcal{P}$ be such that $t(P_i) = p_{\{i\}}$. Then, $f^p(P_i, P_{-i}) \leq p_{\{i\}}$ for each $P_{-i} \in \mathcal{P}^{n-1}$. This implies that $x \in V_i$.

2. $\boldsymbol{x \leq p_{\{i\}} < p_{N\setminus\{i\}}}$. Let $P_i \in \mathcal{P}$ such that $t(P_i) = p_{N\setminus\{i\}}$. Then, $f^p(P_i, P_{-i}) \geq p_{N\setminus\{i\}}$ for each $P_{-i} \in \mathcal{P}^{n-1}$ (because $p_{N\setminus\{i\}} \leq p_S$ for each S such that $i \notin S$). This implies that $x \in V_i$.

In both cases, therefore, $x \in V_i$. $\square$

The dictatorial rules are strategy-proof, tops-only and unanimous, therefore they are generalized median voting schemes. It is easy to see that if the agents i is the dictator, then $p_{\{i\}} = a$ and $p_{N\setminus\{i\}} = b$. Trivially these rules are NOM and then we omit then in the following result.

Theorem 3 *A non-dictatorial generalized median voter scheme $f^p : \mathcal{P}^n \longrightarrow X$ is NOM if and only if, for each $i \in N$,*

(i) $p_{N\setminus\{i\}} \in \{a, a+1\}$ and $p_{\{i\}} = b$, or

(ii) $p_{N\setminus\{i\}} = a$ *and* $p_{\{i\}} \in \{b-1, b\}$.

Proof. Let f^p be a non-dictatorial generalized median voter scheme.
($\Longleftarrow$) The proof that conditions (i) or (ii) imply that f^p is NOM follows a similar argument to that of the proof of Theorem 2 and therefore it is omitted.
($\Longrightarrow$) Assume that f^p is NOM. First, assume there is an agent $i^\star \in N$ such that $p_{\{i^\star\}} < p_{N\setminus\{i^\star\}}$. Then, by Lemma 2 , $V_{i^\star} = X$. By Theorem 1, $SV_{i^\star} = X$. Thus, agent $i^\star$ is a dictator, contradicting that f^p is non-dictatorial. Therefore $p_{N\setminus\{i\}} \leq p_{\{i\}}$ for each $i \in N$. Now, the proof follows a similar argument to the proof of Theorem 2 and, therefore, it is omitted. $\square$

4.2 Voting by Committees

Now assume that agents have to choose a *set of objects* from a set K (with $|K| \geq 2$). Then, in this case, $X = 2^K$ and elements of X are subsets of K. A generic element of K is denoted by k. Barberà et al. (1991) characterize, on the restricted domain of separable preferences, the family of all strategy-proof social choice functions satisfying unanimity as the class of voting by committees. A preference P_i of agent i is separable if the division between good objects (those $x \in X$ such that $\{x\}P_i\emptyset$) and bad objects (those $x \in X$ such that $\emptyset P_i\{x\}$) guides the ordering of subsets in the sense that adding a good object leads to a better set, while adding a bad object leads to a worse set. Formally, an agent i's preference $P_i \in \mathcal{P}$ on 2^K is *separable* if for all $x \in 2^K$ and $k \notin x$,

$$x \cup \{k\}P_i x \text{ if and only if } \{k\}P_i\emptyset.$$

Let $\mathcal{S}$ be the set of all separable preferences on 2^K. Observe that for any separable preference its top is the subset of good objects. That is, for any separable preference $P_i \in \mathcal{S}$,

$$t(P_i) = \{k \in K : \{k\}P_i\emptyset\}.$$

We now define the class of rules known as voting by committees. Let N be a set of agents and $k \in K$ be an object. A committee $\mathcal{W}_k$ for k is a non-empty set of non-empty coalitions (subsets) of N, which satisfies the following monotonicity condition:

$$M \in \mathcal{W}_k \text{ and } M \subseteq M' \text{ imply } M' \in \mathcal{W}_k.$$

A rule $f : \mathcal{P}^n \longrightarrow 2^K$ is a *voting by committees* if for each $k \in K$ there is a committee $\mathcal{W}_k$ such that, for each $P \in \mathcal{P}^n$,

$$k \in f(P) \text{ if and only if } \{i \in N : k \in t(P_i)\} \in \mathcal{W}_k.$$

Observe that voting by committees are very simple. They are tops-only and the selected subset of objects at each preference profile is obtained in a decomposable way, object by object. Barberà et al. (1991) characterize this class when it operates on the separable domain as follows.

Proposition 3 *(Barberà et al., 1991) Rule $f : \mathcal{S}^n \longrightarrow 2^K$ is strategy-proof and unanimous if and only if it is voting by committees.*

Given a committee $\mathcal{W} = \{W_k\}_{k \in K}$, let $f^{\mathcal{W}}$ be its associated voting by committees. Furthermore, and to ease notation, we write $O^{\mathcal{W}}(P_i)$ instead of $O^{f^{\mathcal{W}}}(P_i)$.

Remark 4 *Let $x \in X$ be such that for each $k \in x$ there is $M \in \mathcal{W}_k$ with $i \notin M$ and $\{i\} \notin \mathcal{W}_k$. Now let $P_{-i} \in \mathcal{P}^{n-1}$ be such that $t(P_j) = x$ for each $j \in N \setminus \{i\}$. Then, $f^{\mathcal{W}}(P_i, P_{-i}) = x$ for each $P_i \in \mathcal{P}$. Therefore, $x \in O^{\mathcal{W}}(P_i)$ for each $P_i \in \mathcal{P}$.*

Lemma 3 *Let $f^{\mathcal{W}} : \mathcal{P}^n \longrightarrow 2^K$ be a voting by committees, and let $i \in N$. Then, $x \in V_i$ if and only if there is $k^* \in K$ such that either $k^* \in x$ and $i \in M$ for all $M \in \mathcal{W}_{k^*}$ or $k^* \notin x$ and $\{i\} \in \mathcal{W}_{k^*}$.*[4]

Proof. Assume that $x \in V_i$. Then, there is $P_i \in \mathcal{P}$ such that $x \notin O(P_i)$. Thus, by Remark 4, there is $k^* \in K$ such that $k^* \in x$ and $i \in M$ for each $M \in \mathcal{W}_{k^*}$, or $k^* \notin x$ and $\{i\} \in \mathcal{W}_{k^*}$. Thus, there are two cases to consider:

1. **There is $k^* \in K$ such that $k^* \in x$ and $i \in M$ for each $M \in \mathcal{W}_{k^*}$.** Let $P_i \in \mathcal{P}$ be such that $k^* \notin t(P_i)$. Then, $k^* \notin f(P_i, P_{-i})$ and, therefore, $f(P_i, P_{-i}) \neq x$ for each $P_{-i} \in \mathcal{P}^{n-1}$. Thus, agent i vetos x with P_i. Hence, $x \in V_i$.

2. **There is $k^* \in K$ such that $k^* \notin x$ and $\{i\} \in \mathcal{W}_{k^*}$.** Let $P_i \in \mathcal{P}$ be such that $k^* \in t(P_i)$. Then, $k^* \in f(P_i, P_{-j})$ and, therefore, $f(P_i, P_{-i}) \neq x$ for each $P_{-i} \in \mathcal{P}^{n-1}$. Thus, agent i vetoes x with P_i. Hence, $x \in V_i$.

[4]In the contex of voting by committes, when $i \in M$ for all $M \in \mathcal{W}_{k^*}$, it is said that agent i is a *vetoer* of k^*. Be careful that it is a different notion from our vetoer notion in the present paper.

If $\{i\} \in \mathcal{W}_{k^*}$ it is said that agent agent i is *decisive* at k^*.

□

It is clear that dictatorial voting by committees are NOM. In these rules there is $i \in N$ such that $i \in M$ for all $M \in \mathcal{W}_k$ and $\{i\} \in \mathcal{W}_k$ for all $k \in K$. As these cases are trivial, we omit them in our next theorem.

Theorem 4 *A non-dictatorial voting by committees $f^{\mathcal{W}} : \mathcal{P}^n \longrightarrow 2^K$ is NOM if and only if, for each $k \in K$,*

(i) $\bigcap\limits_{M \in \mathcal{W}_k} M = \emptyset$, *and*

(ii) $|M| \geq 2$ *for all* $M \in \mathcal{W}_k$.

Proof. ($\Longleftarrow$) By Lemma 3, (i) a (ii) imply that $V_i = \emptyset$ for each $i \in N$. Then, by Theorem 1, $f^{\mathcal{W}}$ is NOM.
($\Longrightarrow$) Let $f^{\mathcal{W}}$ be non-dictatorial and NOM. The next claim states that $V_i = \emptyset$ for each $i \in N$.
Claim: Let $i \in N$ and assume that $V_i \neq \emptyset$. Then, $SV_i = V_i$ if and only if i is a dictator.
Assume that $SV_i = V_i$. By Lemma 3, there are two cases to consider:

1. **There is $k^* \in x$ and $i \in M$ for each $M \in \mathcal{W}_{k^*}$.** If there is $k \in K$ such that $\{i\} \notin \mathcal{W}_k$ then, by Lemma 3, $y = \{k^*\} \in V_i$. Now, if P_i is such that $t(P_i) = \{k, k^*\}$, $y = \{k^*\} \in O(P_i)$. Then, i does not strongly veto $y \in V_i$. Thus, $\{i\} \in \mathcal{W}_k$ for all $k \in K$. Now, assume that there is $k \in K$ and $M \in \mathcal{W}_k$ such that $i \notin M$. Then, by Lemma 3, $y = \{k^*, k\} \in V_i$. Now, if P_i is such that $t(P_i) = \{k^*\}$, $y = \{k^*, k\} \in O(P_i)$. Thus, i does not strongly veto $y \in V_i$. Hence, $i \in M$ for each $M \in \mathcal{W}_k$ and for each $k \in K$. Therefore, i is a dictator.

2. **there is $k^* \notin x$ and $\{i\} \in \mathcal{W}_{k^*}$.** If there is $k \in K$ such that $\{i\} \notin \mathcal{W}_k$. Then, by Lemma 3, $y = \varnothing \in V_i$. Now, if P_i is such that $t(P_i) = \{k\}$, $y = \varnothing \in O(P_i)$. Thus, i does not strongly veto $y \in V_i$. Then $\{i\} \in \mathcal{W}_k$ for each $k \in K$. Now, assume that there is $k \in K$ and $M \in \mathcal{W}_k$ such that $i \notin M$. Then, by Lemma 3, $y = \{k\} \in V_i$. Now, if P_i is such that $t(P_i) = \emptyset$, $y = \{k\} \in O(P_i)$. Then, i does not strongly veto $y \in V_i$. Hence, $i \in M$ for each $M \in \mathcal{W}_k$ and for each $k \in K$. Therefore, i is a dictator.

This finishes the proof of the Claim.

In order to finish the proof of the theorem, notice that, therefore, $V_i = \emptyset$ for each $i \in N$. Thus, by Lemma 3, $\bigcap_{M \in \mathcal{W}_k} M = \emptyset$ and $|M| \geq 2$ for all $M \in \mathcal{W}_k$, for all $k \in K$. $\square$

If we add anonymity to Proposition 3 the voting by committee class must be reduced to a relevant subclass of rules which are called voting by quota. A voting by committee is a *voting by quota* if, for each $k \in K$, there is $q^k \geq 1$ such that the associated committee $\mathcal{W}_k$ satisfies that

$$M \in \mathcal{W}_k \text{ if and only if } |M| \geq q^k.$$

Corollary 3 *Let $f^q : \mathcal{P}^n \longrightarrow 2^K$ a voting by quota. Then, f is NOM if and only if $1 < q_k < n$ for each $k \in K$.*

Proof. It follows from Theorem 4. $\square$

Corollary 3 is rather surprising. In general, rules in which quotas are either 1 or n are the most robust to manipulation within all voting by quota from several standpoints (for example, see Arribillaga and Massó, 2017; Fioravanti and Massó, 2022). Our result, in contrast, includes them within obviously manipulable ones.

5 Final Remarks

Following the proof of Theorem 1, it can be seen that the condition $SV_i = V_i$ for each $i \in N$ implies NOM even when top-onliness is removed. Although clearly it is not a necessary condition to get NOM. We state this result in the following theorem.

Theorem 5 *Let $f : \mathcal{P}^n \longrightarrow X$ be a unanimous rule. If $SV_i = V_i$ for each $i \in N$, then f is NOM.*

Now, we present some observations for the case in which X is infinite. If adequate assumptions over the set of preferences are assumed in order that (1) and (2) are well-defined, Theorem 1 is also valid in such context.

For example, in the context of median voting schemes, if X is the interval $[a, b] \subseteq \mathbb{R}$, $f^\alpha : \mathcal{P}^n \longrightarrow X$ is a median voter scheme, and $P_i \in \mathcal{P}$, the option set is given by:

$$O^\alpha(P_i) = \begin{cases} [t(P_i), \alpha_{n-1}] & \text{if } t(P_1) < \alpha_1 \\ [\alpha_1, \alpha_{n-1}] & \text{if } t(P_1) \in [\alpha_1, \alpha_1] \\ [\alpha_1, t(P_i)] & \text{if } t(P_i) > \alpha_{n-1} \end{cases}$$

Then, $O^{\alpha}(P_i)$ is a closed interval. Therefore, if $\mathcal{U}$ is the set of all continuous preferences on $[a, b]$ with a unique top (indifference between no top alternatives are admitted), (1) and (2) in the definition of obvious manipulation are well-defined on $\mathcal{U}$. Thus, in this context, it is not possible that an agent strongly vetoes any alternative. We have the following simple characterization of median voter schemes when $X = [a, b] \subset \mathbb{R}$.

Theorem 6

(i) Let $f^{\alpha} : \mathcal{U}^n \longrightarrow [a, b]$ be a median voter scheme. Then, f^{α} is NOM if and only if $\alpha_1 = a$ and $\alpha_{n-1} = b$.

(ii) Let $f^p : \mathcal{U}^n \longrightarrow [a, b]$ be a non-dictatorial generalized median voter scheme. Then, f^p is NOM if and only if $p_{N \setminus \{i\}} = a$ and $p_{\{i\}} = b$ for each $i \in N$.

References

ARRIBILLAGA, R. P. AND J. MASSÓ (2017): "Comparing voting by committees according to their manipulability," *American Economic Journal: Microeconomics*, 9, 74–107.

AZIZ, H. AND A. LAM (2021): "Obvious Manipulability of Voting Rules," in *International Conference on Algorithmic Decision Theory*, Springer, 179–193.

BARBERÀ, S. (2011): "Strategyproof social choice," *Handbook of social choice and welfare*, 2, 731–831.

BARBERÀ, S., H. SONNENSCHEIN, AND L. ZHOU (1991): "Voting by committees," *Econometrica*, 595–609.

FIORAVANTI, F. AND J. MASSÓ (2022): "False-Name-Proof and Strategy-Proof Voting Rules Under Separable Preferences," *Available at SSRN 4175113*.

GIBBARD, A. (1973): "Manipulation of voting schemes: a general result," *Econometrica*, 587–601.

MOULIN, H. (1980): "On strategy-proofness and single peakedness," *Public Choice*, 35, 437–455.

ORTEGA, J. AND E. SEGAL-HALEVI (2022): "Obvious manipulations in cake-cutting," *Social Choice and Welfare*, 1–20.

PSOMAS, A. AND P. VERMA (2022): "Fair and Efficient Allocations Without Obvious Manipulations," *arXiv preprint arXiv:2206.11143*.

SATTERTHWAITE, M. A. (1975): "Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions," *Journal of Economic Theory*, 10, 187–217.

TROYAN, P. AND T. MORRILL (2020): "Obvious manipulations," *Journal of Economic Theory*, 185, 104970.