Robust equilibria

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August, 2023

Abstract

The concept of *Perfect Bayesian Equilibrium* [*PBE*] refines Bayes-Nash equilibria embodying notions of sequential rationality. However, on one hand it leaves beliefs totally unrestricted for information sets off the equilibrium path. On the other hand, it lacks structural robustness, as even the same game modelled with different extensive forms can have a different set of *PBE*.

The *sequential equilibrium* [*SE*] stands out as a suitable refinement that has more structural properties and gives place to reasonable off-path beliefs. However, *SE* requires different agents using exactly the same perturbations to generate consistent beliefs off equilibrium path, which may be too strict a requirement. Furthermore, testing whether a certain assesment could be a potential *SE* is cumbersome, and involves in principle finding the corresponding sequence of perturbations or completely mixed strategies.

In this paper, we develop a generalization of *SE*, the *robust equilibrium* [*RE*], which refines *PBE* in a less restrictive way, and which computation can be algoritmically carried out solving a relatively simple set of inequalities. Although it can be described as an assesment consistent in a more general way by the use of sequences of perturbations, it can also be characterized by an assortment of nodes according to equilibrium behavior strategies.

1 Introduction

1.1 Equilibrium refinements

The concept of *Perfect Bayesian Equilibrium [PBE]* is a useful refinement for games of imperfect or incomplete information, that have agents making choices in successive information sets. *PBE* refine *BNE* using reasonable beliefs as a rationale for justifiable choices.

It is also possible to strengthen the concept of *PBE*: one of the most frequent reformulations is that of the *Strong Perfect Bayesian Equilibrium [*SPBE*]*: Normally in this strengthening, a *SPBE* is defined as a *PBE* that is also a *Sub-game perfect BNE*.

Both *PBE* and *SPBE* allow for freedom in the way beliefs are deemed reasonable, when they apply to information sets out of the equilibrium path. In this regard, the refinement given by the sequential equilibrium prescribes that beliefs must be consistent in the sense of being the limit of consistent beliefs of purely randomized profiles approaching equilibrium play.

Both *PBE* and *SE* have strengths and weaknesses. The *SE* forces agents to have mutual constraints in the way players could justify deviations from equilibrium. *PBE* leave more freedom (if not too much) on the way beliefs could be built out of the equilibrium path.

1.2 Aim of the paper

The aim of this work is to provide an intermediate refinement between *PBE* and *SE* that could overcome the most obvious drawbacks of the simple *PBE* and grasp the reasonability of off-equilibrium beliefs, without getting into the restrictions that intertwine beliefs in a *SE*.

2 Definitions

2.1 Extensive form of a game

To define a game in extensive form, as usual in the literature define the following elements:

- A set of players:

- A (finite[1]) set $I = \{1, 2, \ldots N\}$ of choosing players.
- The additional player *Nature*, a dummy player that randomizes between actions with pre-fixed probabilities, and serves as a way to introduce randomness, luck, etc.

• A game tree: The tree is composed of:

- A set of decision nodes. A node is an instance in which a player or the Nature takes an action.
- A set of actions. An action goes from one node (the immediate predecessor) to another (the immediate successor). The game tree is always directed, in the sense that a succession relationship can be constructed over the nodes, and this constitutes a strict partial order over the set of nodes.
- A set of final nodes. A node is a final node if it has no immediate successor.
- A labeling of nodes that indicate which player plays at each node.
- A partition of the set of decision nodes into *information sets*, that fulfill:
  - At each information set, each node belongs to the same player.
  - There is a one-to-one correspondence between the set of actions in one node and the one in any other node that belongs to the same information set.
- A vector of payoffs for each player at every final node.
- A decision node, called the root node, that has no predecessor. Each game tree has one and only one root node.

A path (from a node N_1 to another one N_m) is a sequence of nodes $N_1, N_2, N_3, \ldots N_m$ such that each node is direct successor to the preceding one. For any game tree, we must have that for any (non root) node, there is exactly one path from the root to that node. This also implies that no node can be a successor to itself.

In this paper, we will limit our attention to finite games (games that have an extensive form with a finite number of nodes). We will also restrict the analysis to games of perfect recall. A game satisfies perfect recall if every path

[1]Though in principle we could consider games with infinite players, we will deal in this paper exclusively with finite games for a finite number of agents.

intersects any information set at most in one node, and for any two nodes N_i and N_j belonging to an information set N for player h, it must be the case that the corresponding paths from the root to the nodes must cross the same information sets of the player in sequentially the same order, and in each of such information sets the same action must have been played. In other words, no player is allowed to forget something he had known of the game before, or an action he had played before.

2.2 Normal form of a game

A game given in extensive form can also be described in another way, the so called normal form. The normal form of a game involves:

- A set of players $I = \{1, 2, \dots N\}$.
- A set S_i of strategies or contingent plans for every agent i.
- The set of possible (pure) profiles given by $\prod_i^N S_i$
- The payoff function, that indicates for each profile an N-dimensional vector of payoffs for each agent.

2.3 Extensions

There are various extensions of this simple scheme.

- One important extension is that of mixed strategies. The mixing extension of the normal form has each agent mixing over the set of strategies to generate mixes. Here a mixing profile is built with the mixing strategies of each agent.

  We typically refer in this work to mixed strategies as a profile of randomizations for each agent: $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$.

- Another important extension is useful to translate mixing strategies and choices to election of actions on each information node. That is the concept of behavior strategies.

  Each agent h chooses a behavior strategy δ^h that indicates which action to take on each information set.

  For the finite games we are analyzing here, in which perfect recall is assumed, we have this well-known result: for every strategy profile in the mixing extension of the normal form, there is a behavior strategy that yields exactly the same result than the strategy mix, and viceversa.

- When justifying certain choice on an information set, many equilibrium refinements use the concept of belief to justify choices under sequential equilibrium.

  A BELIEF SET for an information set is a vector of probabilities assigned to each node of the information set. This is typically understood as the probability the agent playing in the set assigns of having arrived to each node of the information set.

  A BELIEF SYSTEM is a set built with one belief set for each information set of the game tree.

- An ASSESSMENT $\left(\left\langle\left(\sigma^h\right)_{h=1}^N\right\rangle, (\mu^g)\right)$ of the game is a vector built from two components:
  - a strategy profile $\left\langle\left(\sigma^h\right)_{h=1}^N\right\rangle$, where h indexes the agents.
  - a belief system (μ^g), where g indexes the information sets.

3 A motivational example

3.1 An entry game

Let us introduce the following example: Player 1, a potential entrant into a market, must take the decision of whether to enter [E] or not [N].

If he decides entering the market, he must simultaneously interact with another company, Player 2, the incumbent. Both firms must simultaneously choose between fighting [F] and getting into a price war, or accommodate [A] into a friendly competition scheme.

If firm 1 decides against entering, firm 1 earns 0, while the incumbent gets 3.

Payoffs from the simultaneous interaction are given in the matrix of table 1.

In passing, notice that the entrant has A as a dominant strategy in the simultaneous game, while the incumbent has different best responses for each action player 1 could play in this simultaneous stage.

		Incumbent	
		F	A
Entrant	F	$(-3,0)$	$(1,-1)$
	A	$(-2,0)$	$(3,2)$

Table 1: Payoffs in the simultaneous part of the game.

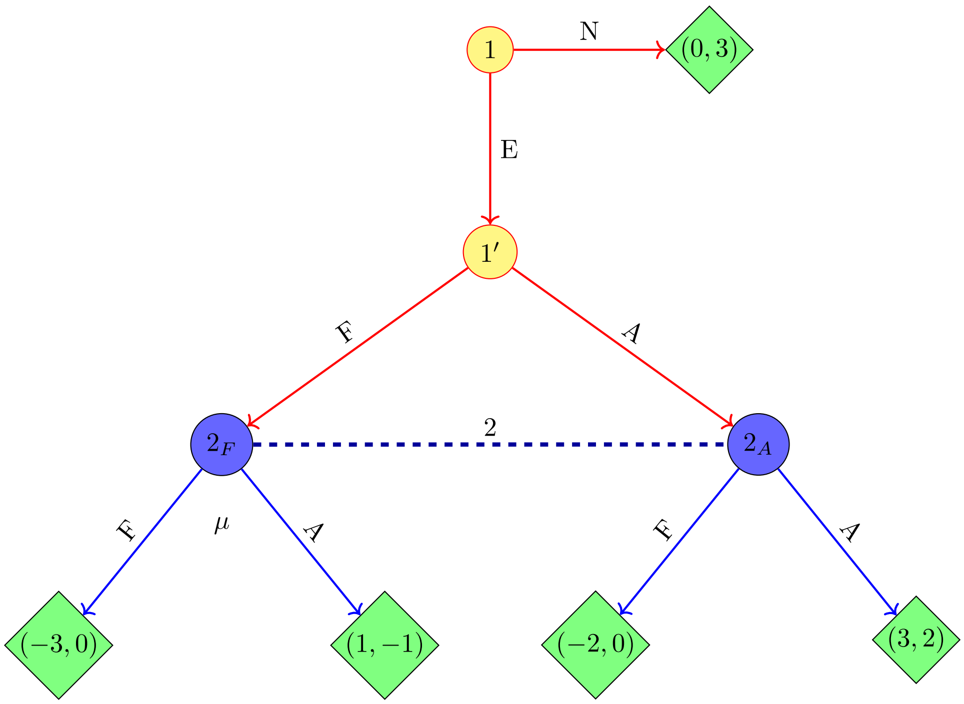

Figure 1: First tree for the entry game.

3.2 Game I

In figure 1, we have modeled an extensive form of the complete game. Player 1 has an initial decision node signalling the E, N decision. After that, if he plays E, he must chose again between F and A after node $1'$; this decision in this case is simultaneously being taken by agent 2 in his information set.

3.3 Normal form

To find the *BNE* we resort to the normal form of the game. We could represent this in matrix form (see table 2).

		Incumbent	
		F	A
Entrant	NF	$(\mathbf{0}, \mathbf{3})$	$(0, 3)$
	NA	$(\mathbf{0}, \mathbf{3})$	$(0, 3)$
	EF	$(-3, 0)$	$(1, -1)$
	EA	$(-2, 0)$	$(\mathbf{3}, \mathbf{2})$

Table 2: Normal form of the entry game.

3.4 Bayesian Nash equilibria

3.4.1 Pure equilibria

In the table 2 above, we have marked best responses for each agent; it is clear that the game has 3 pure *BNE*: profiles $\langle NF, F\rangle$, $\langle NA, F\rangle$ and $\langle EA, A\rangle$.

3.4.2 Mixed equilibria

We won't discuss mixed equilibria in depth. We just remark that agent 1's EF is strictly dominated by EA. Discarding EF, agent 2 finds F weakly dominated. We can then deduce that the only non-degenerated mixed equilibrium is $\langle (p, 1-p, 0, 0), (q, 1-q)\rangle$ with $q \geq {}^3/_5$.

3.4.3 (Pure) equilibrium characterization

The equilibrium profile $\langle EA, A\rangle$ is in some sense the most "sensible". Agent 1 plays E, and in the following simultaneous sub-game, both players choose Nash equilibrium actions $\langle A, A\rangle$. The other two profiles are more inconsistent in one way or the other.

Profile $\langle NF, F\rangle$ is highly problematic: it has agent 1 playing F, a dominated action in the simultaneous part of the game.

On the other hand, profile $\langle NA, F\rangle$ is more appealing: for a start, it makes agent 1 play the dominant strategy A.

To distinguish these 3 equilibrium profiles, we will delve into a common refinement in the literature, that of *PBE*.

3.5 Perfect equilibria

The most widespread definition of Perfect Bayesian Equilibrium [*PBE*] is the following:

Let $\left(\left\langle\left(\sigma^h\right)_{h=1}^N\right\rangle, (\mu^g)\right)$ be an assessment of an extensive form of a game. Then, this assesment constitutes a *PBE* if:

i) the profile $\left\langle\left(\sigma^h\right)_{h=1}^N\right\rangle$ constitutes a BNE for the game.

ii) Each agent, under his beliefs, is maximizing his expected payoff with the behavior strategy followed on each information set [Sequential rationality].

iii) Beliefs are updated with the use of Bayes' rule whenever it is possible [Consistency].

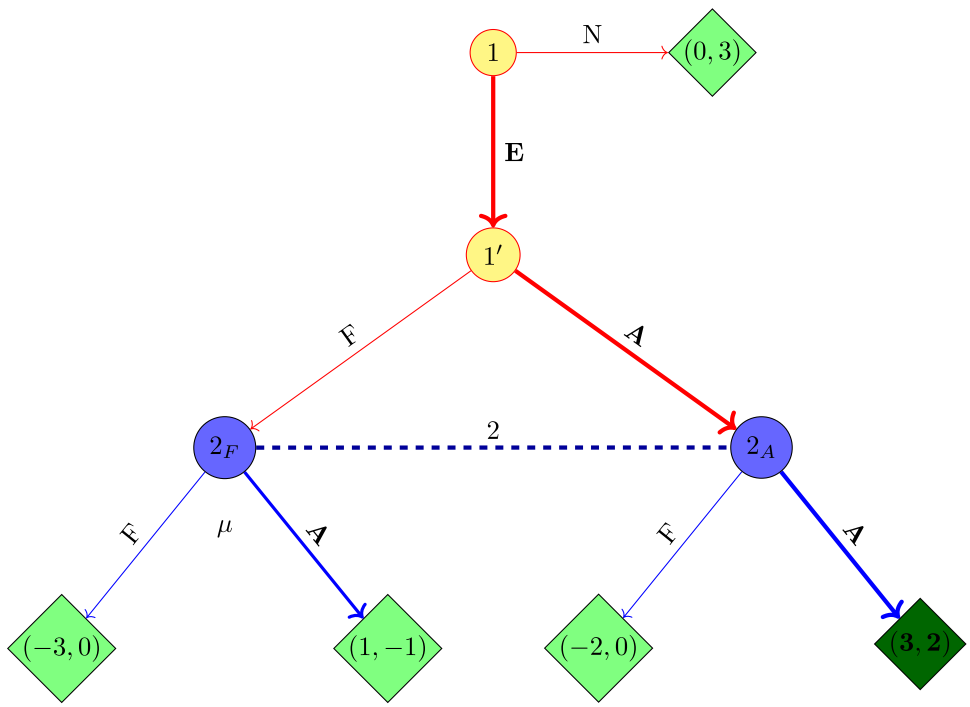

Figure 2: The equilibrium $\langle EA, A\rangle$ marked in the game tree.

3.5.1 $\langle EA, A\rangle$

Figure 2 shows the chosen actions under the profile $\langle EA, A\rangle$ on each information set.

We could acknowledge that this profile fits the definition of *EBP*: each agent could be seen as maximizing payoffs on each information set in which he must play; in particular, agent 2, which has an information set with more than one node, must set a belief set over this information set. However, Bayes' rule forces such belief set: $\mu = \Pr(2F|2)$ and $1 - \mu = \Pr(2A|2)$ are forced to be 0 and 1, respectively. Under $\mu = 0$, agent 2 clearly maximizes his payoffs with the chosen action played in this equilibrium.

3.5.2 $\langle NF, F \rangle$

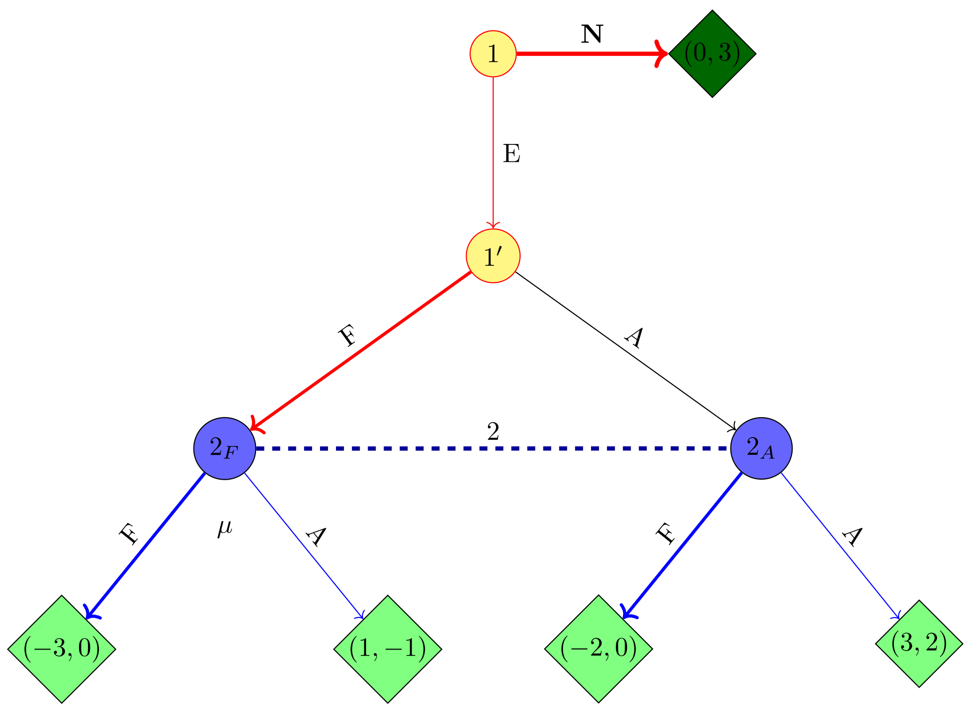

Figure 3: Equilibrium profile $\langle NF, F \rangle$ on the game tree.

The profile $\langle NF, F \rangle$ (See figure 3) is clearly problematical. We could understand agent 2's option of playing F with any belief $\mu \geq 2/3$. However, agent 1 is not satisfying sequential rationality on his information set $1'$. Thus, this profile cannot be a *PBE*.

3.5.3 $\langle NA, F \rangle$

In figure 4 we could see this profile marked in the presented game tree. $\langle NA, F \rangle$ is a *PBE*: Player 1 maximizes payoffs on each information set (node) in which he has to make a choice, subject to what is being played. Player 2 chooses in his

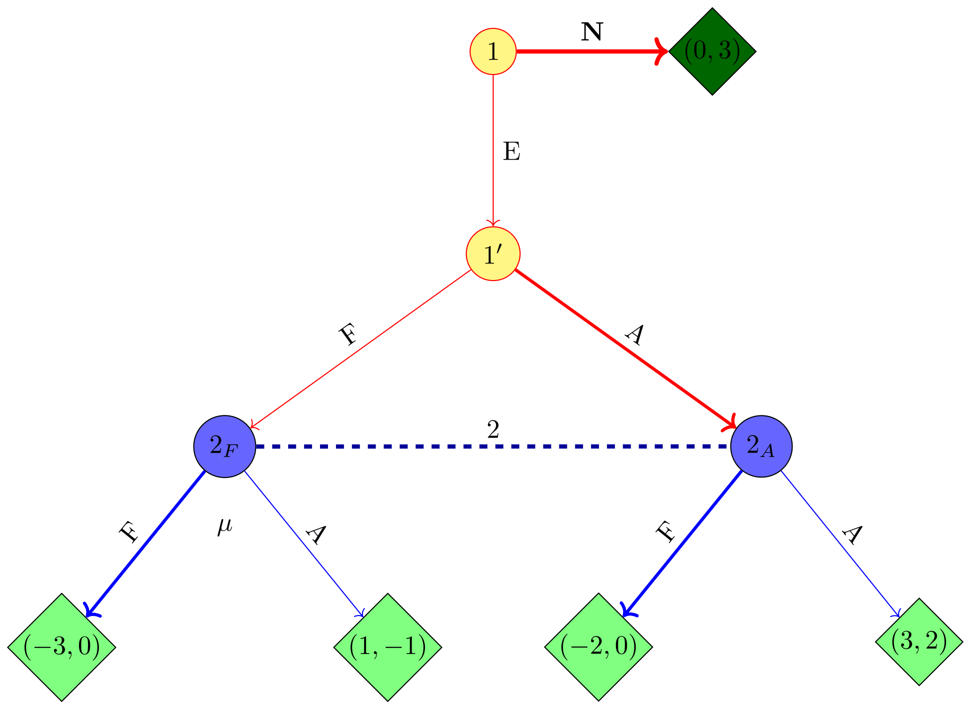

Figure 4: Equilibrium profile $\langle NA, F\rangle$ on the game tree.

information set an action compatible with logical beliefs, given the equilibrium profile: with $\mu = \Pr(2F|2) \geq {}^2/_3$ action F is justified, and although agent 1 could be playing A from node $1'$, such node is not reached so as to enforce updating of μ by means of Bayes' rule so that it falls out of the interval $[{}^2/_3, 1]$.

4 Difficulties

Of the 3 pure *BNE*, we have seen one $[\langle NF, F\rangle]$ that can be discarded as some player is not maximizing payouts on some information set. Another one $[\langle EA, A\rangle]$ does not present major issues. The third one $[\langle NA, F\rangle]$ presents an equilibrium under which each agent is maximizing payoffs given logical, consistent beliefs.

However, we notice an interesting problem with this equilibrium: after agent 1's choice between E and N, players interact under a simultaneous instance where both agents choose between F and A. The game tree of figure 1 is one possible way of modelling the interaction.

4.1 Second tree

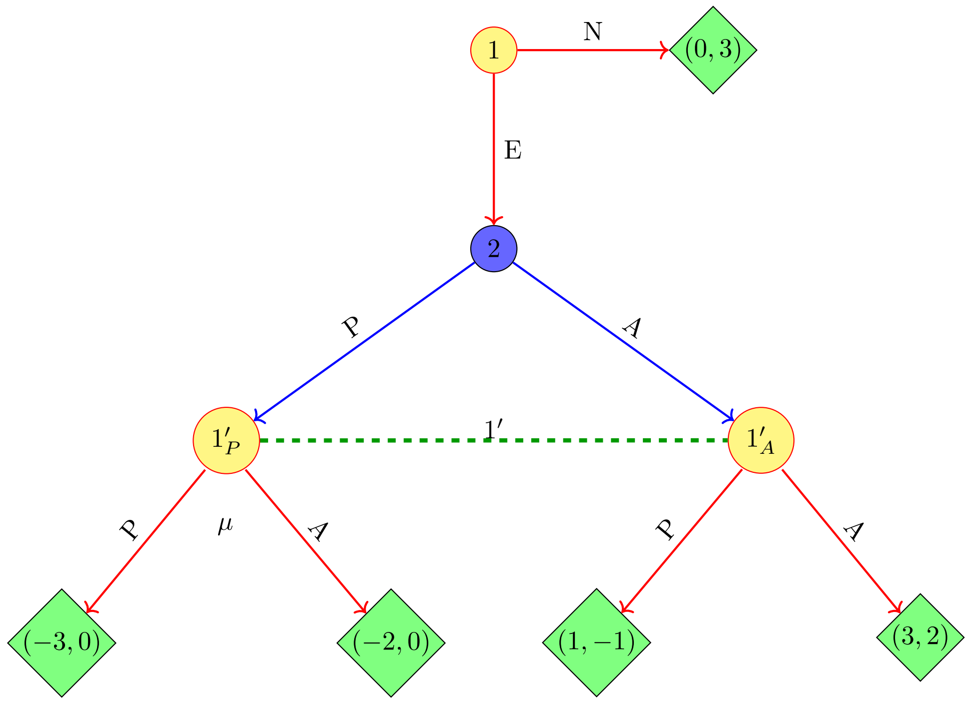

Figure 5: Another possible modellization of the entry game.

However, such game form is not unique. In fact, figure 5 shows another possible extensive form for exactly the same game.

In this extensive form, player 2 has a node immediately following decision node 1 after the first agent chooses E. Then, player 1 chooses under imperfect information, without knowing agent 2's choice.

Both extensive forms are equivalent, in terms of the normal structure of the game, and simply differ by a modelling decision of the analyst. We should not expect many substantive changes in the central aspects of the game.

4.1.1 ***BNE***

Since both extensive forms have exactly the same normal form (the one expressed in the matrix of table 2) the set of *BNE* must be the same for both. Thus, the pure *BNE* for the game tree of figure 5 are still $\langle EA, A\rangle$, $\langle NP, P\rangle$ and $\langle NA, P\rangle$.

4.1.2 *PBE*

We would thus be tempted to conclude that the *PBE* corresponding to the game tree of figure 5 are the same that the ones seen on §3.5: after all, the difference between both extensive forms is just a matter of modelling decision.

Alas, it could be seen that this is not the case here! Neither $\langle NF, F \rangle$ nor $\langle NA, F \rangle$ are *PBE*. The first profile cannot sustain sequential rationality for player 1 at information set $1'$ under no beliefs, since A gives greater payoffs for any possible behavior strategy of player 2. The second profile cannot be sustained as *PBE* because even when player 1 is choosing the payoff-maximizing action A, player 2 is not maximizing payoffs with F, since A would make him better-off.

4.1.3 Discussion

In the first place, this result is quite unappealing: depending on a choice of modelling, we would have a perfect equilibrium profile or not!

A reason why this happens can be understood considering the role that beliefs take in the choice of each agent. In the first modellization, we have agent 2 as "guardian" of beliefs. Under this extensive form, we could sustain F as a reasonable choice, since this agent does not have a dominant strategy in the simultaneous part.

In the second modellization, we make the first player bear the weight of beliefs, but this player has only one sequentially rational alternative: choosing A. Under this logic, player 2 cannot be sequentially rational playing F.

Another thing to consider is that the concept of *PBE* conditions every behavior on information sets posterior to an agent's decision, but it does not conditions immediate previous choices [except for nodes on the equilibrium path]. Why do we ask in figure 4 that agent 1 maximizes payoffs regarding what agent 2 chooses after him, but do not ask agent 2 to have beliefs compatible with what player 1 is immediately choosing before? This may be more logical in games of perfect information, but it is not that clear in a more general environment.

We would like some more robustness in the way equilibrium refinements deal with different ways of modelling a game. Clearly, *EBP* is weak in this sense.

4.2 Alternatives

4.2.1 strong *PBE*

A *BNE* is a STRONG PERFECT BAYESIAN EQUILIBRIUM [*SPBE*] if the equilibrium profile is a *PBE* and at the same time is sub-game perfect.

We could see that none of the profiles $\langle NA, P\rangle$ o $\langle NP, P\rangle$ are subgame-perfect: hence, this stronger refinement discards these problematical equilibria.

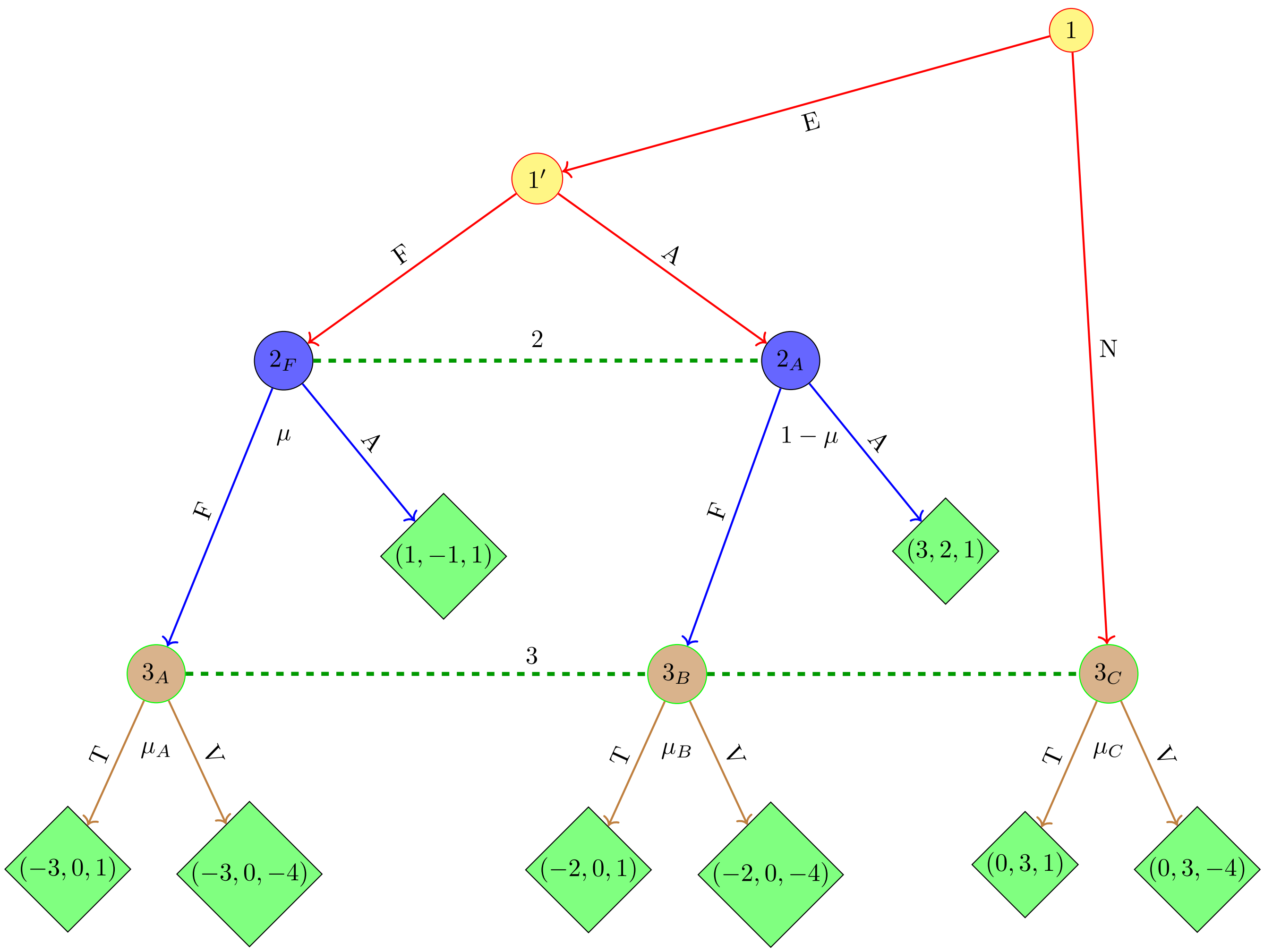

Figure 6: Entry game modified tree, without strict subgames.

Notice however the game tree in figure 6. In this modified version of the entry game, in which a third player makes a trivial choice between take or veto payments, we find three equilibria homologous to the previous ones: $\langle EA, A, T\rangle$; $\langle NA, P, T\rangle$; and $\langle NP, P, T\rangle$. On the other hand, there is a new *BNE* $\langle EA, A, V\rangle$ similar to $\langle EA, A, T\rangle$ but having agent 3 vetting in case game play falls out of $\langle EA, A\rangle$'s path.

We observamos that in this modelling, profile $\langle NA, P, T\rangle$ is a *PBE* with beliefs $\mu \geq 2/3$, and $\mu_A = \mu_B = 0, \mu_C = 1$. However, if we model the tree again changing the game order putting first agent 2, we have the game tree in figure 7. Under this modellization, equilibrium profile $\langle NA, P, T\rangle$ is not a *PBE* [and thus, not a *SPBE* either]: player 2 is not maximizing his payoffs in the node in which he plays, since A is the choice that generates greater payoffs.

4.2.2 Sequential equilibrium

One of the most used refinements in the literature is that of SEQUENTIAL EQUILIBRIUM [SE]: this concept involves, as the *PBE*, the analysis of assessments.

If we take up the tree of figure 4, we will notice that the *EBP* $\langle NA, P\rangle$ is not a *SE*: as figure 8 shows, any assessment must include the consistent belief $\mu = 0$.

The SE is an interesting refinement. We could think in two complex points of this concept:

- On one hand, deviations from equilibrium under consideration by each agent must be under certain relation.
- On the other hand, the definition of SE demands the definition of randomizations near equilibrium, or so called perturbations. This is easily studied for a simple tree, but it gets complicated easily with bigger trees.
- Finally, up to what we know, there is no generic mechanism or algorithmthat allows direct computation of the *SE* of a game.

The SE has certain points of critique. Consider the following game, given in extensive form in figure 9:

The game is given in normal form in table 3.

The game has only one pure *BNE*: the profile $\langle \gamma, b, II\rangle$. In a mixed equilibrium, player 1 always plays γ. There is a set of BNE with $\langle (\sigma_\alpha, \sigma_\beta, \sigma_\gamma), (\sigma_a, \sigma_b), (\sigma_I, \sigma_{II})\rangle = \langle (0, 0, 1), (q, 1-q), (r, 1-r)\rangle$ satisfying $q \leq 3/4$ and $r \leq \frac{1}{2(1-q)}$.

Since any path through 2_α or 2_β is off-equilibrium, any set of beliefs for the information set 2 (and 3) could be consistent with the application of Bayes' rule,

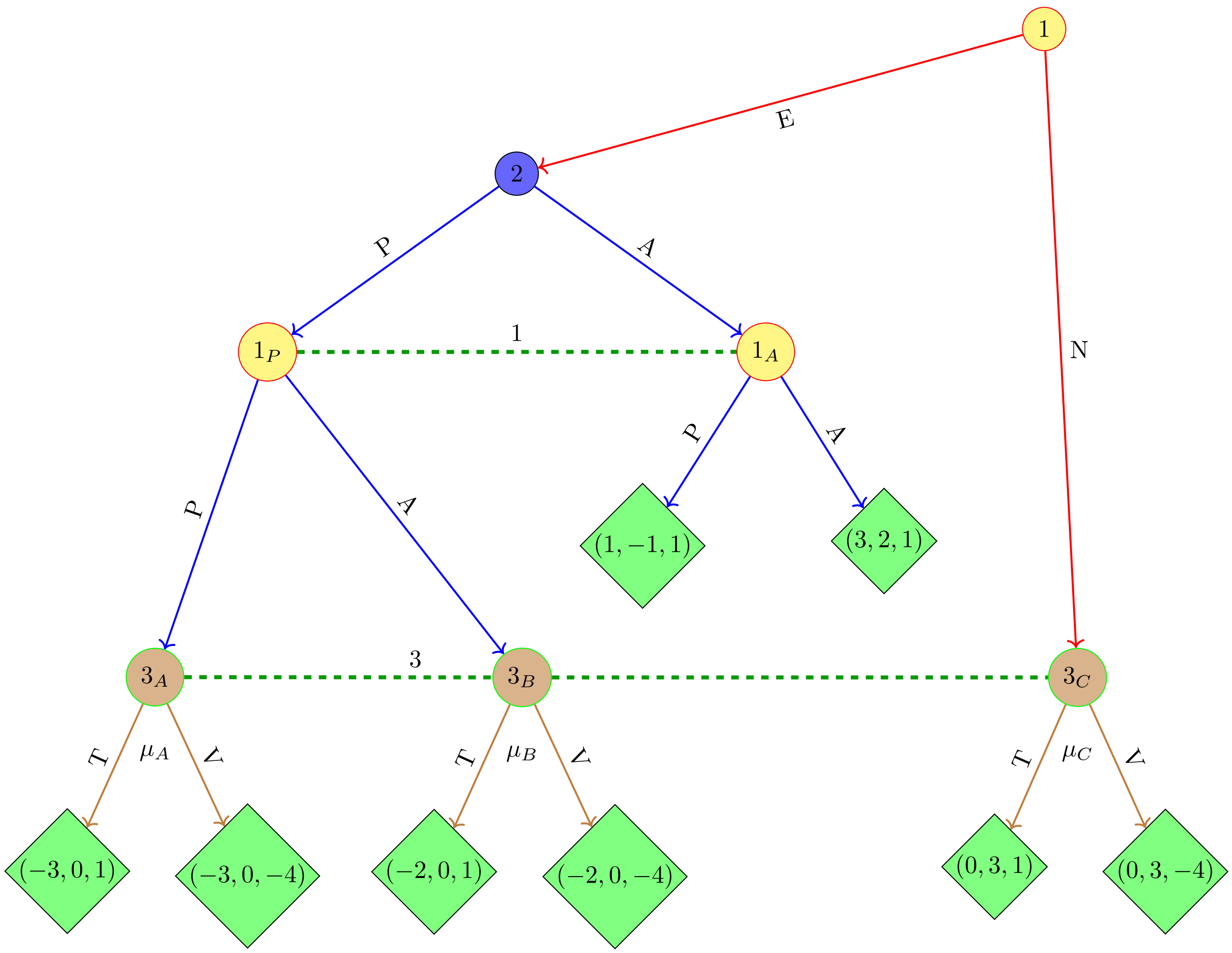

Figure 7: Entry game modified tree, without strict subgames, with another order of nodes.

so that an assessment with a set of beliefs that fulfills sequential rationality is guaranteed to constitute a *PBE*.

What about the *SE* of the game? The *SE* is usually described as a refinement in which agents tend to "believe the game will drift off-equilibrium path in a similar way", but that is nearly an overstatement.

Maybe find a quote?

Figure 10 shows the complete set of mixed *BNE* (and *PBE*, provided accurate beliefs are set) for game 2. Highlighted are two segments over the randomization space (q, r) that form *SE* assessments with the correct beliefs.

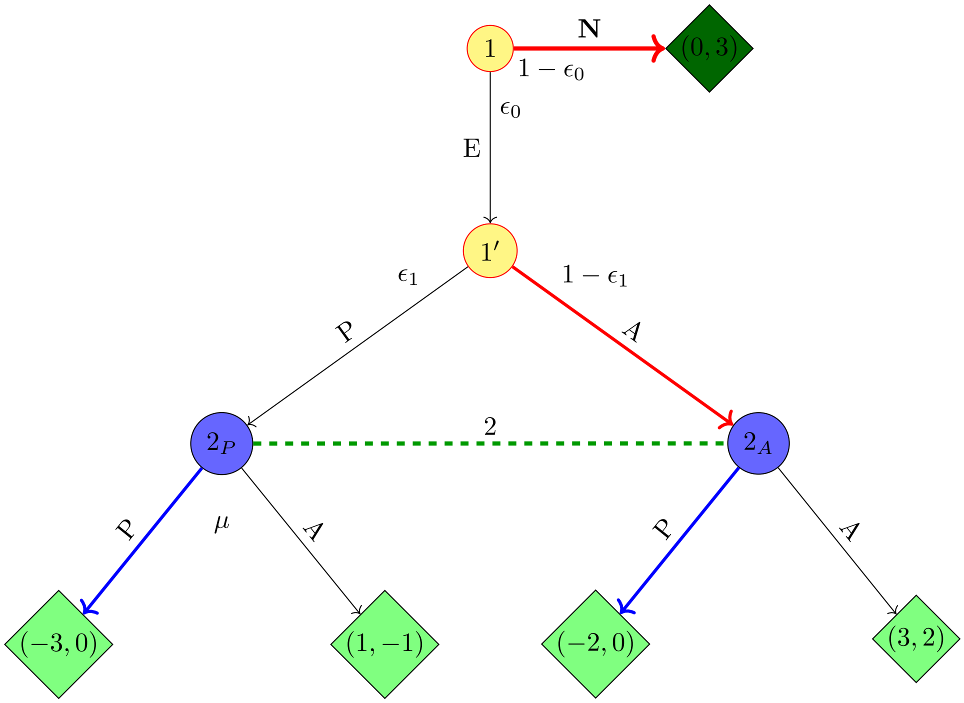

Figure 8: $\langle NA, P\rangle$ no es un equilibrio secuencial.

Player 3

I — Player 2

Player 1		a	b
	α	$(2, 2, -1)$	$(4, -1, 0)$
	β	$(4, 0, 5)$	$(0, 1, 3)$
	γ	$(3, 2, 2)$	$(3, 2, 2)$

II — Player 2

Player 1		a	b
	α	$(2, 2, -1)$	$(2, 0, 0)$
	β	$(4, 1, 1)$	$(0, 1, 3)$
	γ	$(3, 2, 2)$	$(\mathbf{3}, \mathbf{2}, \mathbf{2})$

Table 3: Normal form for the 2nd example game.

Let us consider the pure equilibrium profile $\langle \gamma, b, II\rangle$: under beliefs $\mu = \Pr(2_\alpha|2), \nu = \Pr(3_i|3)$, an assessment $(\langle \gamma, b, II\rangle, (\mu, \nu))$ is a *SE* if $\mu = 0$ and $\nu = 1$. Notice that this involves a completely different evaluation about off-equilibrium play: agent 2 believes that if he has to play, it is because agent 1 has deviated with β, while agent 3 believes that if he has to make a choice, it is because agent 1 has played α. They couldn't disagree more!

In fact, this disagreement is found in every *SE*, except for the small red

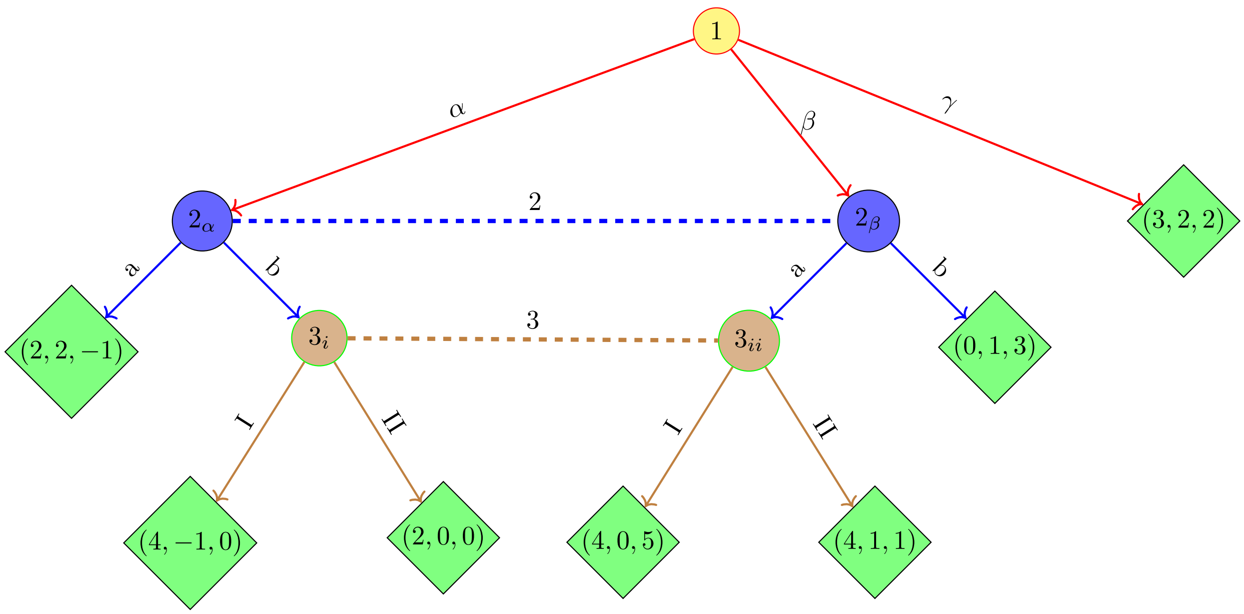

Figure 9: Extensive form for a second example.

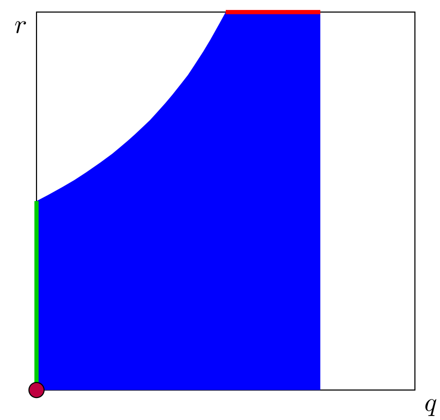

Figure 10: The space of possible values of q and r in a mixed equilibrium for game 2, with *SE* areas highlighted.

segment in figure 10.

If agents differ on hypothetical game play, why is this assessment consistent? Basically because there is a common set of perturbations of gameplay that allows in the limit for the said beliefs. For example, taking perturbations $\sigma_\alpha = \frac{1}{n^2}$, $\sigma_\beta = \frac{1}{n}$ and $\sigma_a = \frac{1}{n^2}$, we approach the said beliefs as $n \to \infty$.

The approach of using consistency by means of totally mixed randomizations is sensible, in the sense that it discards illogical or unnatural beliefs.

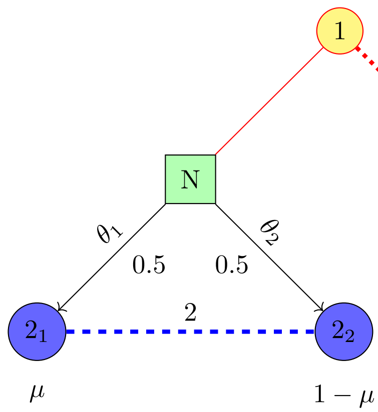

Figure 11: A fragment of a tree.

In figure 11, in this section of a game player 1 plays away from *Nature* and player 2's information set. A *PBE* would admit any possible value of μ to form the beliefs of agent 2 on his information set. However, every *SE* will have $\mu = 1/2$, the probability alloted by Nature to each type or move θ_1, θ_2.

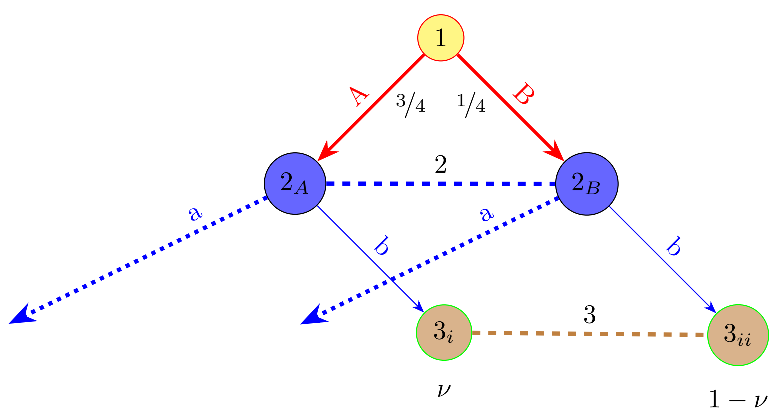

Figure 12: A fragment of another tree.

In figure 12, we start the game with agent 1, who mixes between A and B with probabilities $(3/4, 1/4)$. Then game proceeds to player 2's information set, in which the agent plays a with certainty. A *PBE* could in principle allow for any value of ν that conforms to sequential rationality, while agent 2's beliefs would be chained to agent 1's mixing. However, a *SE* could only be formed with an assessment satisfying $\nu = 3/4$, which is reasonable, since any other belief would imply an infringement of Bayes's rule or a violation of the structure of the game.

Such restrictions to agents' beliefs are sensible. We will prove in this paper however that the approach of consistency taken by the *SE* paradigm is not necessarily the only possibility of refining the *PBE* concept, and that indeed a generalization of the *SE* can be proved to be enough to guarantee such sensible belief constrains.

5 Robust equilibrium

5.1 Definitions

To advance with our formulation, we make some previous definitions. We bear in mind a game in extensive form, as developed in 2.1.

Definition 5.1. The COMMON ORIGIN for a group of nodes $\{N_i\}_{i \in I}$ is the nearest common ancestor or predecessor, that is:

- a node O that is predecessor to every N_i and;
- if there is a node M that is predecessor to every node N_i, then O is successor to M.

We will denote the operator that returns the common origin of two nodes N_1 and N_2 as $\mathcal{O}(N_1, N_2)$.

Example 5.2. In the modified game shown in figure 6, nodes 3_A and 3_B have node $1'$ as its common origin, while nodes 3_B and 3_C have 1 as common origin. In the extensive form of figure 7, nodes 3_A and 3_B have node 1_P as its common origin.

As we could see in figure 12, the fact that an action is played[2] or not in an information set like the one for player 2, regarding beliefs of player 3 on his information set, is irrelevant for the purposes of our refinement. We then approach a series of definitions that will help us deal with such actions.

[2]When we say that an action is played or assigned positive probability under a profile, we are meaning that the behavior strategy derived from the strategy profile assigns positive probability to the action. A similar notion is intented when we say that an action happens with zero probability under a profile. This is regardless of whether the action is isolated from the equilibrium path.

Definition 5.3. We say that nodes N_k and N_l form a BRIDGE [with routing action a] for nodes N_i and N_j if:

i) N_k and N_l belong to the same information set.

ii) N_k and N_l are (without loss of generality) predecessors to N_i and N_j, respectively.

iii) The same action a advances to the respective nodes from the predecessors[3].

Example 5.4. In the game section shown in figure 12, nodes 2_A and 2_B form a *bridge* for nodes 3_i and 3_{ii}.

Example 5.5. In the game shown in extensive form in figure 9, nodes 2_α and 2_β do not form a *bridge* for nodes 3_i and 3_{ii}, since different actions (a and b) go down the path to such nodes.

If nodes N_k and N_l form a *bridge* for nodes N_i and N_j with routing action a, we say that under some game profile $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$ the bridge is:

- *open*, if action a is never played under the profile.
- *closed*, if action a carries positive probability under the profile.

Definition 5.6. a RELEVANT ACTION from node O to a successor node N_i (in regard to another node N_j) is an action that takes place in the path from O to N_i (that is, an action going from O or a successor of O, to N_i or a predeccessor of N_i) and is not a routing action for a bridge between N_i and N_j.

In other words, a relevant action happens on the path from O to N_i, and there is no homologous action happening on N_j's path from the origin.

Example 5.7. In figure 9, b is a relevant action from 1 to 3_i, in regard to 3_{ii}.

Example 5.8. In figure 12, b is not a relevant action from 1 to 3_i, in regard to 3_{ii}: an homologous action b happens between 2_B and 3_{ii}. Action B, on the other hand, is a relevant action from 1 to 3_{ii} in regard to 3_i.

Definition 5.9. Under a game profile $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$ a DETOUR from node O to node N_i (in regard to node N_j) is a relevant action a_i from node O to node N_i (in regard to node N_j) that is assigned under $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$ zero probability.

[3] That is to say, if action a advances in the path to N_i from N_k, then the homologous action goes from N_l to N_j or an predecessor of N_j.

Example 5.10. Consider the game given in extensive form in figure 9. Under the mixed *BNE* $\langle (0,0,1), (q, 1-q), (r, 1-r) \rangle$ with $q = r = 1/2$, in which both agent 2 and 3 equally ramdomize between their two strategies, α is a detour from root to 3_i in regard to 3_{ii}, and β is a detour from 1 to 3_{ii} in regard to 3_i. Under the (pure) *BNE* $\langle (0,0,1), (0,1), (0,1) \rangle$, action a is a detour from the root node to 3_{ii} in regard to 3_i: a is a relevant action that agent 2 never plays following his strategy under this equilibrium profile.

We now finish this section with the following tools:

Definition 5.11. With N_k a predecessor to node N_i, and the play profile $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$, the probability WEIGHT function

$$p\left(N_i, N_k, \left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle\right) \tag{1}$$

is given by the product of the weights or probabilities of each action in the path from N_k to N_i, given by the profile.

Example 5.12. As can be deduced with the help of figure 2, under the *BNE* $\langle EA, A \rangle$ in the entry game, we have

$$p\left(2_F, 1, \langle EA, A \rangle\right) = 0 \tag{2}$$

and

$$p\left(2_A, 1, \langle EA, A \rangle\right) = 1 \tag{3}$$

Example 5.13. From the fragment of extensive form shown in figure 12, it is apparent that under the equilibrium profile

$$p\left(2_B, 1, \langle (3/4, 1/4), (1, 0), \ldots \rangle\right) = 1/4 \tag{4}$$

and

$$p\left(3_{ii}, 1, \langle (3/4, 1/4), (1, 0), \ldots \rangle\right) = 0 \tag{5}$$

When comparing the respective weights for a pair of nodes that have a shared open bridge, it will be convenient to have a modified version of the weight function that would determine the relative weights the nodes would have were the routing action played. To this end, we define a modified version of the precedent function:

Definition 5.14. With N_k a predecessor to node N_i, and the play profile $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$, in regard to another node N_j, considering that

$$p\left(N_i, N_k, \left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle\right) = \prod_{l=1}^{g} \sigma_l\left(a_l\right) \tag{6}$$

... where $a_1, a_2 \dots a_g$ are the successive actions taken in the path from N_k to N_i, and $\sigma_l\left(a_l\right)$ is the probability or weight given to action a_l under the behavior strategy equivalent to the strategy profile $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$, the probability BRIDGE-ADJUSTED WEIGHT function is given by

$$\tilde{p}\left(N_i, N_k, \left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle, N_j\right) = \prod_{l=1}^{g} \tilde{\sigma}_l\left(a_l\right) \tag{7}$$

... where $\tilde{\sigma}_l\left(a_l\right)$ is given by $\sigma_l\left(a_l\right)$, unless action a_l is the routing action for an *open bridge* between N_i and N_j, in which case we set $\tilde{\sigma}_l\left(a_l\right) = 1$.

Example 5.15. In the game for which a fragment is shown in figure 12

$$p\left(3_{ii}, 1, \left\langle \left(3/4, 1/4\right), \left(1, 0\right), \dots \right\rangle\right) = 0 \qquad \text{as in (5)} \tag{8}$$

but

$$\tilde{p}\left(3_{ii}, 1, \left\langle \left(3/4, 1/4\right), \left(1, 0\right), \dots \right\rangle, 3_i\right) = 1/4 \tag{9}$$

Notice that

$$\frac{\tilde{p}\left(3_{ii}, 1, \left\langle \left(3/4, 1/4\right), \left(1, 0\right), \dots \right\rangle, 3_i\right)}{\tilde{p}\left(3_{i}, 1, \left\langle \left(3/4, 1/4\right), \left(1, 0\right), \dots \right\rangle, 3_{ii}\right)} = \frac{1/4}{3/4} = \frac{1}{3} \tag{10}$$

... which is precisely the ratio of probabilities one would obtain in a consistent belief under a *SE* for this game.

5.2 Principles for robustness

As commented above, our aim is to develop an equilibrium refinement that restricts beliefs to fulfill some logical requirements. Thus, we propose the following

Proposition 5.1. *The sought refinement must satisfy the following propositions:*

a) It must follow gameplay whenever possible.

b) It must use Bayes' rule whenever it is admissible.

c) When comparing belief probabilities for two nodes of an information set off equilibrium path, we must assume that somehow the common origin of those nodes is reached. If admissible, Bayes' rule must be taken into account for play past the common origin.

d) The effect of open bridges on reasonable beliefs must be taken into account, as explained when we analyze the fragment of tree in figure 12.

5.3 Classification of nodes

5.3.1 Some technical results

Based on the previous discussion, we are now fitted for the task of developing the sought refinement. First, we will define some useful relationships between nodes:

Definition 5.16. Let N_i and N_j be two nodes in the extensive form of a game. Let $O = \mathcal{O}(N_i, N_j)$ be its common origin.

We say that N_i is ISOLATED from N_j (under a game profile $\left\langle \left(\sigma^h\right)_{h=1}^{N} \right\rangle$) if there is a *detour* from O to N_i in regard to N_j.

Example 5.17. According to the fragment of game shown in figure 11, neither node 2_1 is isolated from 2_2, nor 2_2 is isolated from 2_1. This is true for any possible game profile, since any such profile has Nature actions θ_1 and θ_1 assigned positive probabilities.

Example 5.18. In the extensive form of the entry game given in figure 5, node $1'_A$ is isolated from node $1'_P$ under pure profiles $\langle NF, F\rangle$ and $\langle NA, F\rangle$, but not under $\langle EA, A\rangle$.

Notice that some of the developed concepts depend only on the tree structure (common origin, bridge) and some others also depend on the strategy profile taken in reference (detours, isolation, etc.).

Definition 5.19. Let us consider two nodes N_i and N_j with common origin O. We will say that, under certain strategy profile $\left\langle \left(\sigma^h\right)_{h=1}^{N} \right\rangle$

- N_i is DOMINATED BY N_j, which we will write $N_i \ll N_j$, if N_i is isolated from N_j but N_j is not isolated from N_i. We will also say in this case that N_j DOMINATES N_i, or $N_j \gg N_i$.

- N_i is UNCOUPLED FROM N_j, which we will write $N_i \parallel N_j$, if N_i is isolated from N_j and N_j is isolated from N_i.

- N_i is PROPORTIONAL TO N_j, which we will write $N_i \propto N_j$, if neither N_i is isolated from N_j nor N_j is isolated from N_i.

Example 5.20. Consider the entry game described under the extensive form of figure 1. Under equilibrium profile $\langle NF, F\rangle$, we have $2_A \ll 2_F$. In the other two pure profiles $\langle NA, F\rangle$ and $\langle EA, A\rangle$, we have $2_A \gg 2_F$.

Under any mixed equilibrium profile for this game, we have $2_A \propto 2_F$.

Lemma 5.2 (Completeness)**.** *Let N_i and N_j be two nodes. Then, either $N_i \parallel N_j$, or $N_i \propto N_j$, or $N_i \ll N_j$, or $N_i \gg N_j$.*

Proof. Trivial, for the isolated and not isolated characters are mutually exclusive, and the definitions from 5.19 cover all possible cases. □

Lemma 5.3 (Proportionality factor)**.** *Let N_i and N_j be two nodes. Let $O = \mathcal{O}(N_i, N_j)$ its common origin. If under certain profile $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$, $N_i \propto N_j$, then the adjusted weights*

$$\tilde{p}\left(N_i, O, \left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle, N_j\right) \tag{11}$$

and

$$\tilde{p}\left(N_j, O, \left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle, N_i\right) \tag{12}$$

are always greater than zero.

Proof. None of these 2 values could be zero, because in that case we would have one node isolated from the other, which would contradict the fact that $N_i \propto N_j$. □

Corollary. *For any pair of nodes N_i, N_j such that under certain profile $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$,*

$N_i \propto N_j$, the ratios

$$\kappa_{i,j} = \frac{\tilde{p}\left(N_i, \mathcal{O}\left(N_i, N_j\right), \left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle, N_j\right)}{\tilde{p}\left(N_j, \mathcal{O}\left(N_i, N_j\right), \left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle, N_i\right)} \tag{13}$$

and its reciprocal

$$\kappa_{j,i} = \frac{\tilde{p}\left(N_j, \mathcal{O}\left(N_i, N_j\right), \left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle, N_i\right)}{\tilde{p}\left(N_i, \mathcal{O}\left(N_i, N_j\right), \left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle, N_j\right)} \tag{14}$$

are always well-defined.

Lemma 5.4. *Let N_i, N_j and N_k three nodes. Let $O_m = \mathcal{O}\left(N_i, N_j\right)$ and $O_n = \mathcal{O}\left(N_j, N_k\right)$ be the indicated common origins. Then, either O_m is predecessor to O_n, or O_n is predecessor to O_m, or $O_m = O_n$. Furthermore, $\mathcal{O}\left(N_i, N_k\right)$ is O_m or O_n.*

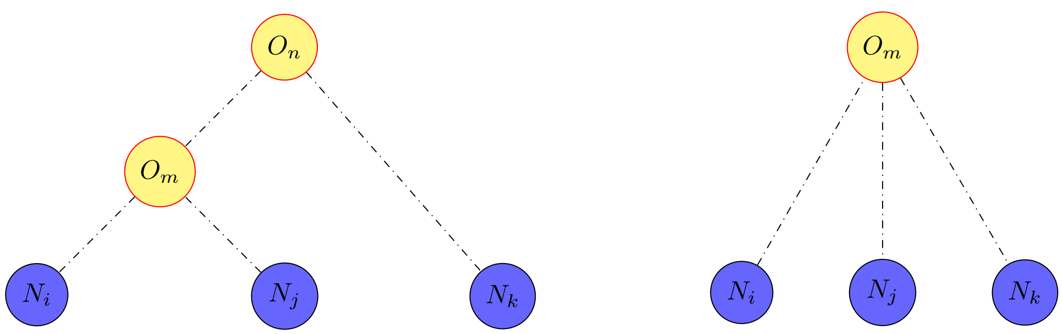

Figure 13: Schematic forms of the cases indicated in lemma 5.4.

Proof. By definition, both O_m amd O_n are predecessors to N_j. By the structure of the extensive form of the game, O_m and O_n must comply with the first part of the thesis, that is, one must be the predecessor of the other, if not the same node.

Now, let $O_o = \mathcal{O}\left(N_i, N_k\right)$. If $O_m \neq O_n$, let us assume without loss of generality that O_n precedes O_m. Then, it must be $O_o = O_n$. This can be seen easily with the help of figure 13. A similar idea applies to the case $O_n = O_m$. □

Lemma 5.5. *Fixed a certain strategy profile, being $\ll$, $\propto$ and $\|$ binary relations over nodes of a game tree, we have:*

i) $\ll$ *is an asymmetric, irreflexive relation: for every pair of nodes* N_i *and* N_j, $\neg(N_i \ll N_i)$ *and* $N_i \ll N_j \implies \neg(N_j \ll N_i)$.

ii) $\propto$ *and* $\|$ *are reflexive, symmetric relations: for every pair of nodes* N_i *and* N_j, $N_i \propto N_i \wedge N_i \| N_i$*;* $N_i \propto N_j \implies N_j \propto N_i$*; and* $N_i \| N_j \implies N_j \| N_i$.

Proof. Trivial. □

Theorem 5.6 (Transitivity)**.** *Let* N_i*,* N_j *and* N_k *three nodes of the game tree. Then,*

i) $N_i \ll N_j \wedge N_j \ll N_k \implies N_i \ll N_k$ *[*$TRANS_1$*].*

ii) $N_i \propto N_j \wedge N_j \propto N_k \implies N_i \propto N_k$ *[*$TRANS_2$*].*

Proof.

I) $N_i \ll N_j \wedge N_j \ll N_k \implies N_i \ll N_k$

a) Suppose $O_o = O_n$, predecessor to O_m. That's exactly the case described in the left part of figure 13.

Since $N_j \ll N_k$, there exists a detour from O_n to N_j in regard to N_k. Were this action in the path from O_n to O_m, then cleary N_i would be isolated from N_k. But if this action comes from a node between O_m and N_j, since N_j is not isolated from N_i, this action must come from a bridge between N_i and N_j. But this bridge cannot be extended to N_k, or the action would not be the supposed detour. Summing up, N_i must be isolated from N_k.

On the other hand, if any action is played with zero probability between O_n and N_k, the fact that $N_j \ll N_k$ forces that action to be a routing action of a bridge between N_j and N_k. In every possible case that action must also be included in a bridge that makes it not a detour of N_i in regard to N_k. Thus, N_k cannot be isolated from N_i.

b) Now, let us suppose $O_m = O_o$ and O_o is predecessor to O_n.

In this case, since $N_j \ll N_k$, there exists a detour to N_j in regard to N_k. This action must take place between O_n and N_j. Since N_j cannot be isolated from N_i, this must be a routing action in an open bridge between N_i and N_j, which cannot be extended to N_k by hypothesis (the action is a detour in regard to N_k). Thus, again we deduce N_i is isolated from N_k.

On the other hand, if any action is played with zero probability in the path from O_m to N_k, neither could it be played from O_m to N_i [since $N_i \ll N_j$]. Thus, N_k cannot be isolated from N_i.

c) Finally, if $O_m = O_o = O_n$, the fact that $N_j \ll N_k$ implies that there is a detour from O_o to N_j in regard to N_k, an action that also cannot be played in the path from O_o to N_i, since $N_i \ll N_j$. Thus, N_i must be isolated from N_k.

On the other hand, if there is some action not played in the path from A_o to N_k, it also cannot be played in the way from A_o to N_i, since $N_j \ll N_k$ and $N_i \ll N_j$. Thus, N_k cannot be isolated from N_i.

II) $N_i \propto N_j \wedge N_j \propto N_k \implies N_i \propto N_k$

This second part of the proof is similar to the previous discussion; we only must show here that the hypotheses imply that neither N_i is isolated from N_k, nor the other way around.

□

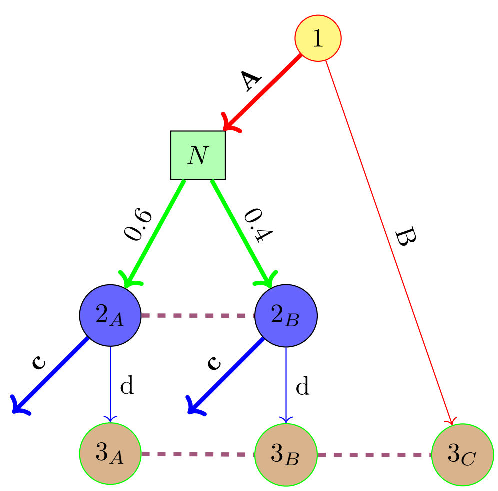

Figure 14: Part of a game tree that shows an example of violation of transitivity by $\|$.

Notice that the relation $\|$ does not have to fulfill transitivity, as the example of figure 14 shows: in that game, agent 1 plays A, Nature mixes between two alternatives and agent 2 plays action c in his information set. We have $3_A \parallel 3_C$ and $3_C \parallel 3_B$. However, $\neg(3_A \parallel 3_B)$; rather $3_A \propto 3_B$.

Corollary. *The relation $\ll$ [and $\gg$] constitute strict partial orders. The relation $\propto$ constitutes an equivalence.*

Theorem 5.7 (Absorption)**.** *Let N_i, N_j and N_k three nodes of an extensive form of a game. Then:*

i) $N_i \propto N_j \wedge N_j \ll N_k \implies N_i \ll N_k$ *[ABS_1].*

ii) $N_i \ll N_j \wedge N_j \propto N_k \implies N_i \ll N_k$ *[ABS_2].*

iii) $N_i \propto N_j \wedge N_j \parallel N_k \implies N_i \parallel N_k$. *[$ABS_3$].*

Proof.

i) Let $O_m = \mathcal{O}(N_i, N_j)$, $O_n = \mathcal{O}(N_j, N_k)$, and $O_o = \mathcal{O}(N_i, N_k)$ the respective common origins.

Again, by 5.4, either $O_o = O_m = O_n$ or $O_o \in \{O_m, O_n\}$. We will approach each case separately.

a) Let $O_o = O_n$, predecessor to O_m, as in the left part of figure 13.

Clearly, $N_j \ll N_k$ implies there is a detour from O_n to N_j in regard to N_k. Were that action in the path from $O_o = O_n$ to O_m, then it clearly will be a detour from O_o to N_i. On the contrary, if this action takes place in the path from O_m to N_j, the fact that $N_j \propto N_i$ would also imply that it must be present in an open bridge between N_i and N_j.

Either way, we have that N_i must be isolated from N_k.

Now, suppose there is an action that is played with zero probability in the path from O_n to N_k. Since $N_j \ll N_k$, this must be a routing action in a bridge between N_j and N_k. If the homologous action takes place in the way from O_n to O_m, this open bridge constitutes also a bridge between N_i and N_k. Were that homologous action in the way from O_m to N_j, then the bridge must be extended to an homologous action between O_m and N_i, since $N_j \propto N_i$. Either way, N_k cannot be isolated from N_i.

b) Let now $O_o = O_m$, predecessor to O_n.

Since $N_j \ll N_k$, there must be a detour from O_n to N_j. This action must be a routing action of an open bridge between N_j and N_i, since $N_j \propto N_i$, so there is a homologous action in the path from O_m to N_i that is played with zero probability, and N_i is isolated from N_k.

On the other hand, let us suppose that there is an action that is played with zero probability in the path from O_m to N_k. If this action happens between O_m and O_n, then it must be the routing action of a bridge between N_i and N_j, as $N_i \propto N_j$. Now, if this action happens between O_n and N_j, it must be the routing action of a bridge between N_k and N_j (since $N_j \ll N_k$), and this bridge must be extended to N_i (since $N_j \propto N_i$). Thus, N_k cannot be isolated from N_i.

c) Finally, let us consider the case $O_o = O_m = O_n$. This case is the shown in the right part of figure 13.

The fact that $N_j \ll N_k$ implies that there is a detour from O_o to N_j in regard to N_k. since $N_i \propto N_j$, there must be a homologous action in the path from O_o to N_i that constitutes a detour from O_o to N_i in regard to N_k. Thus, N_i must be isolated from N_k.

Now, if an action in the way from O_o to N_k is played with zero probability, it must be a routing action to a bridge between N_j and N_k, since $N_j \ll N_k$. But this bridge must also be extended to N_i, since otherwise there would be a detour to N_j in regard to N_i, which would violate $N_i \propto N_j$. Thus, N_k cannot be isolated from N_i.

ii) Secondly, $[ABS_2]$ is proved following a similar argument to the preceding. One must show that N_i must be isolated from N_k, and that N_k cannot be isolated from N_i, using the fact that $N_i \ll N_j$ and $N_j \propto N_k$.

iii) Finally, to show that $N_i \propto N_j \wedge N_j \parallel N_k \implies N_i \parallel N_k$, we will use both preceding results in a proof by contradiction.

Let us suppose that $\neg (N_i \parallel N_k)$. Then, by lemma 5.2, either $N_i \propto N_k$, $N_i \ll N_k$, or $N_i \gg N_k$.

a) Suppose $N_i \propto N_k$. Then, $N_k \propto N_i$ and $N_i \propto N_j$ implies by transitivity (theorem 5.6) that $N_k \propto N_j$, contradicting $N_j \parallel N_k$.

b) Suppose now $N_i \ll N_k$. Since $N_j \propto N_i$, this would imply by $[ABS_1]$ that $N_j \ll N_k$, again contradicting $N_j \parallel N_k$.

c) Finally, suppose $N_i \gg N_k$. Then, $N_k \ll N_i$ and $N_i \propto N_j$ imply by $[ABS_2]$ that $N_k \ll N_j$, contradicting $N_j \parallel N_k$ again.

We could conclude then that $N_i \parallel N_k$ must hold.

□

5.3.2 Applications to nodes of an information set

The preceding results can be applied to any set of nodes. In particular, taking into account our aims, it will be productive to apply those to the nodes in an information set.

In the following, let $M = \{N_1, N_2, \dots N_l\}$ be the (finite) set of l nodes in an information set of a game in extensive form, with[4] $l > 1$. We will assume certain strategy profile[5] $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$.

Definition 5.21. Let M_{undom} be the set of maximal elements for $\gg$, that is, the set of nodes $\{N_g \in M | \nexists N_i \in M, N_i \gg N_g\}$. We call this the set of UNDOMINATED NODES of M.

Let $M_{dom} = M \setminus M_{undom}$. We call the elements of M_{dom} the DOMINATED NODES of M.

Lemma 5.8. *For any non-empty[6] set M, M_{undom} is non-empty.*

Proof. The lemma is trivial for a singleton. If M has more than one element, the result follows from completeness [lemma 5.2] and the asymmetric character of $\ll$ [lemma 5.5]. □

Example 5.22. Consider the extensive form for the entry game in figure 1. If $M = \{2_F, 2_A\}$, the information set for agent 2 in this extensive form, then:

- Under the pure equilibrium profiles $\langle EA, A \rangle$ and $\langle NA, F \rangle$, $M_{undom} = \{2_A\}$.

- Under the pure equilibrium profile $\langle NF, F \rangle$, $M_{undom} = \{2_F\}$.

[4] From our perspective, information sets that are singletons are completely trivial.

[5] It is apparent that the classification of nodes depend on the played profile; in other words, whether a node dominates another, or is uncoupled from another one is dependent on the considered game play.

[6] We were actually focused on the case $\#M > 1$, but this result is particularly also true for singletons.

- Under any other (mixed) equilibrium profile, $M_{undom} = \{2_F, 2_A\} = M$. Thus, in this case $M_{dom} = \emptyset$.

Example 5.23. Let $M = \{3_i, 3_{ii}\}$ the nodes in the information set of agent 3 in the game of figure 9. Under any *BNE*, $3_i \parallel 3_{ii}$; thus, $M_{undom} = M$ for any profile.

Theorem 5.9. *Assume certain strategy profile. Then:*

A) M, M_{undom} and M_{dom} can be partitioned into equivalence classes according to $\propto$, such that for every pair of nodes N_i, N_j that belong to the same class, one has $N_i \propto N_j$.

B) In the preceding partition fo M, no class has elements of both M_{undom} and M_{dom}.

Proof. Trivial. Partition in equivalence classes follows from the fact that $\propto$ is an equivalence. The fact that no class could have elements of both M_{undom} and M_{dom} is derived from the transitivity of $\propto$ and the exclusive character of $\propto$ and $\ll$. □

Definition 5.24. Let M be an information set. Fixing a certain strategy profile $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$, let us consider M_{undom} as above.

If M_{undom} has only one equivalence class by $\propto$, then we say that M is TRIVIAL[7] [under $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$]. Otherwise, we say that M is NON-TRIVIAL [under $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$].

Example 5.25. Let M be the information set of player 2 in the extensive form in figure 1. Here, M is *trivial* for any equilibrium profile: under any pure *BNE*, M_{undom} is a singleton. Under any other (mixed) equilibrium, $M_{undom} = M$, and $2_F \propto 2_A$.

Example 5.26. The information set for agent 2 in the game partly shown in figure 11 is *trivial* for any possible strategy profile: we have $2_1 \propto 2_2$ by force.

Example 5.27. In the game described in extensive form in figure 6, the profile $\langle EA, A, \bullet \rangle$ generates a non-trivial information set for agent 3. Here, $3_A \ll 3_B$ and $3_B \parallel 3_C$.

[7] The logic of this terminology will be clearer with the subsequent discussion.

Definition 5.28. Let M be some non-trivial information set under a profile $\left\langle\left(\sigma^h\right)_{h=1}^N\right\rangle$.

Let $C_1, C_2, \ldots C_p \subseteq M_{undom}$ the equivalence classes of M_{undom} under $\propto$. Notice that it is always possible[8] to pick a set of p representative nodes $N_1 \in C_1, N_2 \in C_2, \ldots N_p \in C_p$. Indeed, we are able to perform this selection for any *non-trivial* information set M.

Considering such selection, we will call $N_1, N_2, \ldots N_p$ the FREE NODES of M [under $\left\langle\left(\sigma^h\right)_{h=1}^N\right\rangle$].

Example 5.29. Following example 5.27, 3_B and 3_C can be taken to be the *free nodes* of agent 3's information set.

Free nodes are a relevant issue for non-trivial information sets. These only happen off profile gameplay, as the next result shows.

Theorem 5.10. *Let $\left\langle\left(\sigma^h\right)_{h=1}^N\right\rangle$ be certain strategy profile [most of the times we will use this result with equilibrium profiles]. If the information set M is on profile gameplay [on equilibrium path, if the profile is an equilibrium], that is, if under the profile any node of M is reached with positive probability, then M is* trivial.

Proof. Suppose the information set $M = \{N_1, N_2, \ldots N_l\}$ is on profile gameplay. Then, at least one node N_k of M is reached with positive probability. The path that goes from the root node to N_k does not have an action with null weight. Any node of M which is not reached with positive probability must then be dominated by N_k. We show now that every node of M that is played with positive probability belongs to the same equivalence class under $\propto$ of N_k. This is indeed the case. Supose N_j is also reached with positive probability. Then, any action in the path from the root node to N_j is played with positive weight. That means that $N_k \propto N_j$, since there is no detour possible for any other relation between nodes. □

5.4 Robust beliefs

Definition 5.30. Let us fix a profile $\left\langle\left(\sigma^h\right)_{h=1}^N\right\rangle$. Let M be an information set of the game, with nodes $N_1, \ldots N_l$. A belief set for M is an l-uple $(\mu_1, \mu_2, \ldots \mu_l)$. We say that the belief set $(\mu_i)_{i=1}^l$ is ROBUST if:

[8] Indeed, since every M is finite, we do not even have to resort to the axiom of choice to guarantee that.

I) $\sum_{i=1}^{l} \mu_i = 1$ [trivial requirement].

II) $\forall i,\ \mu_i \geq 0$ [trivial requirement].

III) $N_h \in M_{dom} \implies \mu_h = 0$ [dominance requirement].

IV) $N_i \propto N_j \implies \mu_i = \kappa_{i,j}\mu_j$ with $\kappa_{i,j}$ as in (13) [proportionality requirement[9]].

Example 5.31. In the entry game shown in extensive form in figure 1, the assessment formed by the profile $\langle NA, F\rangle$ together with beliefs given by $\mu_{2_F} = q, \mu_{2_A} = 1 - q$ with $q \geq 2/3$ constitutes a *PBE*, as was discussed in §3.5.3.

In this case, the belief set for agent 2 in his information set is not robust: as implied in example 5.22, $2_F \ll 2_A$, so under robust beliefs, condition III) demands $\mu_{2_F} = 0$.

Robust belief sets have constrains regarding proportional and dominated nodes. They have free hand for fixing beliefs for free nodes.

Lemma 5.11. *Let $\left\langle \left(\sigma^h\right)_{h=1}^{N} \right\rangle$ be a strategy profile. Let $M = \{N_1, N_2, \ldots N_l\}$ be an information set. Consider two robust belief sets*

$$\mu = (\mu_1, \mu_2, \ldots \mu_l) \tag{15}$$

and

$$\nu = (\nu_1, \nu_2, \ldots \nu_l) \tag{16}$$

for M. Then, μ and ν must assign the same probabilities to dominated nodes [zero]. Furthermore, if nodes N_i and N_j fulfill $N_i \propto N_j$, then $\mu_i = \kappa_{i,j}\mu_j$ and $\nu_i = \kappa_{i,j}\nu_j$.

Proof. Trivial. □

Example 5.32. For the information set 3 in figure 14, any belief set satisfying

$$\mu_{3_A} \cdot 0.4 = \mu_{3_B} \cdot 0.6 \tag{17}$$

$$\mu_{3_A} + \mu_{3_B} + \mu_{3_C} = 1 \tag{18}$$

will be *robust.*

[9]If both $\mu_i \neq 0, \mu_j \neq 0$, IV) could also be written in the more natural form $\frac{\mu_i}{\mu_j} = \kappa_{i,j}$. This presentation allows for null belief values for such nodes.

Now, we present an important characterization of such robust beliefs in the spirit of conistency and perturbations:

Theorem 5.12. *Let* Γ *be certain game; and* $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$ *be certain strategy profile of a game. Let* M *be an information set of the game, with nodes* $N_1, \ldots N_l$*. Let us consider belief sets* $(\mu_1, \mu_2, \ldots \mu_l)$ *for* M*.*

I) Suppose $\left\{ \left\langle \left(\sigma_n^h\right)_{h=1}^N \right\rangle \right\}_{n=1}^\infty$ *is a sequence of totally mixed strategy profiles (we could also speak of the equivalent behavior strategies) that has* $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$ *as limit. Let* $(\mu_{1n}, \mu_{2n}, \ldots \mu_{ln})$ *be the belief set for* M *for each term of the sequence, that is derived according to Bayes' rule [this set is unique for each term of the sequence, since each profile is totally mixed]. If*

$$\mu = (\mu_1, \mu_2, \ldots \mu_l) = \lim_{n \to \infty} (\mu_{1n}, \mu_{2n}, \ldots \mu_{ln}) \tag{19}$$

then $(\mu_1, \mu_2, \ldots \mu_l)$ *form a* robust *set of beliefs.*

II) Let $(\mu_1, \mu_2, \ldots \mu_l)$ *be a robust set of beliefs for* M*. Then, there exists a sequence of totally mixed strategy profiles* $\left\{ \left\langle \left(\sigma_n^h\right)_{h=1}^N \right\rangle \right\}_{n=1}^\infty$ *with belief sets for* M $(\mu_{1n}, \mu_{2n}, \ldots \mu_{ln})$ *derived from Bayes' rule for each term of the sequence, such that:*

- $\lim_{n \to \infty} \left\{ \left\langle \left(\sigma_n^h\right)_{h=1}^N \right\rangle \right\}_n = \left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$.
- $\lim_{n \to \infty} (\mu_{1n}, \mu_{2n}, \ldots \mu_{ln}) = (\mu_1, \mu_2, \ldots \mu_l)$

Sketch of proof.

Part I) is rather trivial. Let $O = \mathcal{O}(N_1, \ldots N_l)$. It is easy to see that for any term of the sequence, and any pair of nodes N_i, N_j:

$$\frac{\mu_i}{\mu_j} = \kappa_{i,j}. \tag{20}$$

If $N_i \in M_{dom}$ and $N_j \in M_{undom}$, such $\kappa_{i,j}$ tends to zero as $n \to \infty$. That implies that for any dominated node N_i, we must have $\mu_i = 0$. A similar argument can be used for the case of proportional undominated relations.

Part II) can be sketched in the following way: firstly, we may limit ourselves to consider perturbartions of the form $\frac{a_t}{n^{b_t}}$ for detours.

Now, let M_F the set of free nodes, as defined above. We only have to show that it is always possible to find a family of perturbations for a certain set of belief values for the selected free nodes, since proportionality and dominance constrains will follow, because according to I) any family of perturbations generate robust belief sets in the limit.

To show the required condition, it suffices to consider that M_F consists of nodes uncoupled with one another. Thus, we will be able to set aside one action for each node in M_F such that the corresponding beliefs are set. One must be specially careful with the case of a free node that must be assigned zero probability in the belief set. $\square$

Example 5.33. Taking up the extensive form from example 5.29, under the (equilibrium) profile $\langle EA, A, T\rangle$, we have $M_F = \{3_B, 3_C\}$.

Thus, according to the definition of robust beliefs [$\rightarrow$ definition 5.30], any belief set $(\mu_{3_A}, \mu_{3_B}, \mu_{3_C})$ with $\mu_{3_A} = 0$ will be robust.

It can be seen that any such belief set can be dealt with employing perturbations for actions F [from 2_A to 3_B] and N [from 1 to 3_C]. For instance, perturbations $\phi_F = \frac{1}{n^2}, \phi_N = \frac{1}{n}$ work for $(\mu_{3_A}, \mu_{3_B}, \mu_{3_C}) = (0, 0, 1)$; $\phi_F = \frac{1}{n}, \phi_N = \frac{1}{n^2}$ avail us to deal with $(\mu_{3_A}, \mu_{3_B}, \mu_{3_C}) = (0, 1, 0)$; and $\phi_F = \frac{a}{n}, \phi_N = \frac{b}{n}$ can deal with any belief set $(\mu_{3_A}, \mu_{3_B}, \mu_{3_C}) = (0, a, b)$ for $a > 0, b > 0$.

Example 5.34. For the entry game shown in extensive form in figure 1, under the (mixed) profile $\langle (p, 1-p, 0, 0), (q, 1-q)\rangle$, the only robust set of beliefs for agent 2 in his information set is given by $(\mu_{2_F}, \mu_{2_A}) = (p, 1-p)$.

5.5 Robust equilibrium

Having defined robust beliefs, we develop the corresponding refinement we were looking for.

Definition 5.35. A ROBUST EQUILIBRIUM is an assessment $\left(\left\langle \left(\sigma^h\right)_{h=1}^{N} \right\rangle, (\mu^g)\right)$ such that:

A) $\left\langle \left(\sigma^h\right)_{h=1}^{N} \right\rangle$ constitutes a *BNE* for the game.

B) At each information set, the corresponding agent maximizes payoffs according to the beliefs of the assessment.

C) Each belief set of the system of beliefs (that is, each component of (μ^g), one for each information set) is *robust*.

This refinement satisfies precisely the properties demanded in 5.1.

5.6 Relationship between concepts

We finish the section with these final results, that round up the idea mentioned in the introductory part, that of finding a middle ground between *PBE* and *SE*.

Thus, we show that the *RE* is precisely a refinement of *PBE* and a generalization of *SE*.

Lemma 5.13. *Let* $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$ *be an equilibrium profile. Suppose that the assessment* $\left(\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle, (\mu^g)\right)$ *is a robust equilibrium. Then, it is also a* PBE*.*

Proof. A *RE* automatically complies with the restrictions of a *PBE* regarding sequential rationality. A *PBE* does not impose restrictions in information sets off equilibrium path. We just have to check that Bayes' rule is applied over information sets on equilibrium path. But this is indeed the case, as theorem 5.10 implies. □

Theorem 5.14. *Let the assessment* $\left(\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle, (\mu^g)\right)$ *be a* SE*. Then, it is also a* RE*.*

Proof. An assessment is a *SE* only if it is a *BNE* that satisfies sequential rationality and there is a sequence $\left\{\left\langle \left(\sigma_n^h\right)_{h=1}^N \right\rangle\right\}_{n=1}^\infty$ of completely mixed strategy profiles converging to the said equilibrium profile $\left\langle \left(\sigma^h\right)_{h=1}^N \right\rangle$, generating belief systems constistent with Bayes' rule use and also converging to belief system (μ^g).

To show that such assessment is a *RE*, it only remains to show that such belief system is formed by robust belief sets. But that is true, since we could use the first part of theorem 5.12 applying the common mixing sequence $\left\{\left\langle \left(\sigma_n^h\right)_{h=1}^N \right\rangle\right\}_{n=1}^\infty$ to each information set. □

6 Concluding remarks

6.1 Recap

We have developed an equilibrium concept that serves our purposes. It is sufficient to discard problematic equilibria as the profile $\langle NA, F\rangle$ of the entry game.

At the same time, this equilibrium refinement is less stringent on the belief relationship betwen nodes.

Notice in particular that in the game described in extensive form in figure 9, any equilibrium profile has agent 1 playing γ with probability 1.

Thus, we could fix any values to the beliefs of agent 2 and 3 and the corresponding assessment will constitute a *RE*, if beliefs sustain sequential rationality.

One way of thinking about the robust equilibrium is considering it a refinement of PBE that fulfills the demanded principles given in proposition 5.1, extending the use of Bayes' rule whenever we could suppose a common node has been reached.

The other is to consider it a generalization of *SE*, such that instead of restricting the perturbations to be a uniform system for the entire tree, is allowing different perturbations for every node. This could be reasonable, as the example game of figure 9 shows. If agents 2 and 3 have (for instance, under the pure BNE) differing beliefs about the arriving path to their respective information sets, why not letting each agent see consistency of beliefs by means of different mixing sequences?

6.2 Further work and extensions

We believe both RE and SE to be robust under equivalent game representations. To prove this, in the first place we need to formalize different game representations in an abstract framework and show that for any equilibrium in one representation, there is a homologous one in any other.

A very accessible future work is the representation of a RE as just a system of equalities and inequalities: a set of restrictions for equilibrium play; another one for sequential rationality; and finally, another one set of equalities for belief robustness.

Finally, an interesting extension is the development of the relationship between *SE* and *RE* by means of similar equations and relations. Notice that in the game shown in figure 9, there is plenty of room to fix beliefs for both information sets of 2 and 3. To the quoted relations above-mentioned that define a *RE*, we could add a relationship between beliefs over information set 2 and 3 such that we could check if a *RE* is also a *SE*.

Indeed, for that game if we consider for instance the pure *RE*, these relationships are given in the following: $\mu_{2_\alpha} = 0$ or $\mu_{3_{ii}} = 0$.