

Unconditional Quantile Partial Effects via Conditional Quantile Regression*

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Abstract

This paper develops a semi-parametric procedure for estimation of unconditional quantile partial effects using quantile regression coefficients. The estimator is based on an identification result showing that, for continuous covariates, unconditional quantile effects are a weighted average of conditional ones at particular quantile levels that depend on the covariates. We propose a two-step estimator for the unconditional effects where in the first step one estimates a structural quantile regression model, and in the second step a nonparametric regression is applied to the first step coefficients. We establish the asymptotic properties of the estimator, say consistency and asymptotic normality. Monte Carlo simulations show numerical evidence that the estimator has very good finite sample performance and is robust to the selection of bandwidth and kernel. To illustrate the proposed method, we study the canonical application of the Engel's curve, i.e. food expenditures as a share of income.

Keywords: Quantile regression, unconditional quantile regression, nonparametric regression.

JEL: C14, C21.

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1 Introduction

Conditional quantile regression (CQR) is a general approach to estimate conditional quantile partial effects (CQPE), i.e., the effect of a covariate variable of interest (*ceteris paribus*) on the conditional quantile distribution of the outcome.¹ CQR is a useful way to represent heterogeneity using a set of parameters to characterize the entire conditional distribution of an outcome variable given a list of observable covariates.

Recently, unconditional quantile regression (UQR), proposed in a seminal paper by [Firpo, Fortin, and Lemieux \(2009\)](#), has attracted interest in both applied and theoretical literatures. UQR is an important tool for practitioners since it provides a method to evaluate the impact of changes in the distribution of the explanatory variables on quantiles of the unconditional (marginal) distribution of the outcome variable. This method allows researchers to investigate important heterogeneity in the variable of interest. Naturally, UQR leads to the unconditional quantile partial effect (UQPE), which refers to the effect of a covariate (*ceteris paribus*) on the unconditional quantile distribution of the outcome variable.²

[Firpo, Fortin, and Lemieux \(2009\)](#) propose several ways to estimate the UQPE. The most popular approach is the recentered influence function (RIF) regression method, commonly referred to as RIF regression. It is a two-step procedure, where in the first stage one estimates the RIF, and in the second step, a standard OLS regression of the RIF on covariates estimates the UQPE. While the method is appealing due to its simplicity, it relies on ability of the researcher to specify a regression equation for the influence function, a relatively abstract object.

This paper suggests an alternative method for estimation of UQPE using simple CQR. The procedure is based on the identification result of UQPE that explores information contained in the CQPE. [Firpo, Fortin, and Lemieux \(2009, p.959\)](#) show an important theoretical result connecting CQPE and UQPE that, when considering a continuous covariate, the UQPE can be expressed as a weighted average of the CQPE. In fact, the UQPE can be written as the conditional average of the CQPE effects (evaluated where unconditional and conditional quantiles are equal), given the outcome variable (evaluated at the unconditional quantile). Hence, by starting with the common assumption of linearity, a simple reweighting of the CQR coefficients using density functions delivers the UQPE. Derivation of this result relies on a function matching the conditional quantile whose values are the closest to the unconditional quantile. Thus, a useful by-product of the CQR analysis is the ability to express UQPE, for a given quantile $\tau \in (0, 1)$, as a function of CQR.³

¹See, e.g., [Koenker and Bassett \(1978\)](#); [Koenker and Hallock \(2001\)](#); [Koenker \(2005\)](#); [Koenker, Chernozhukov, He, and Peng \(2020\)](#) for comprehensive analyses of CQR.

²Conditional here means that one is conditioning on a set of observable variables, while partial means that one is looking at the effect of one particular covariate controlling for the rest of the covariates. On the other hand, Conditional Quantile Regression (CQR) and Unconditional Quantile Regression (UQR) refer to two regression methodologies to estimate CQPE and UQPE, respectively. Sometimes the acronym of the method is informally interchanged with the parameter of interest, and, therefore, can be somewhat confusing if read lightly.

³Other procedures where statistics of interest are based on a combination of CQR coefficients are the following: [Bera, Galvao, Montes-Rojas, and Park \(2016\)](#) propose to estimate a unique representative CQPE based on an asymmetric Laplace framework, and [Lee \(2021\)](#) considers a general weighted average quantile derivative using the CQR coefficients for a *fixed* quantile level.

We propose a new two-step semi-parametric estimator based on this identification result that employs CQR coefficients to estimate the UQPE. The practical implementation is simple and makes use of the usual practice in empirical research of estimating the CQR process, that is, many quantiles, to explore heterogeneity in conditional effects of a certain covariate. In the first step one uses standard linear QR methods to estimate CQPE from the conditional model of interest over a grid of η -quantiles, and also estimates the unconditional τ -quantile of the outcome of interest. Then, for a given value of the covariates, a matching function can be applied to select coefficients such that unconditional and conditional quantiles are equal. In the second step, one employs a nonparametric regression of the matched CQR coefficients on the outcome, and evaluates this at the unconditional τ -quantile. This is a one-dimensional (reverse) regression: the regressor is the outcome. Mild sufficient conditions are provided for the two-step estimator to have desired asymptotic properties, namely, consistency and asymptotic normality. We derive the convergence rate of the estimator and show that, as expected, it converges at a standard nonparametric rate.

The proposed method offers important advantages over available techniques to compute UQPE. First, our framework provides the researcher with a simple structural conditional quantile model to study, which facilitates the understanding of identification. The conditional QR model allows for a simple and intuitive modelling framework, and it is more familiar to model the main output variable as a function of the covariates using CQR. On the other hand, the effect of covariates on the unconditional model is usually difficult to conceptualize in the existing RIF regression. For instance, it is not obvious how to handle covariates in the data generating process representation for the RIF-OLS and RIF-Logit regressions. There is, thus, a risk of misspecifying the regression in the second stage. In our proposed method, covariates enter the structural CQR in standard way, i.e., in the first step. This approach may be simpler for the researcher since the CQR could be specified using the previous literature or the economic theory. Naturally, the price to be paid is the linearity of the CQR in the first step.

Second, another important motivation and advantage of this paper is to compare and contrast the conditional and unconditional quantile effects. Along these lines, the proposed methods do not require additional modeling assumptions to obtain the UQPE in addition to the CQPE conditions. Since UQPE is nonparametrically identified, the modeling assumptions for both cases would be the same, hence, while the results could still be affected by misspecification of the CQR model, potential differences between UQPE and CQPE results are not be driven by modeling choices.

Finally, there have been considerable extensions on CQR modeling, such as, for example, models for panel data, endogeneity, and large number of covariates (see, e.g., [Koenker, Chernozhukov, He, and Peng \(2020\)](#)). Extensions of the proposed UQPE methods to encompass these situations might be simpler.

Although the literature on applications of UQR methods is extensive, the literature on theoretical developments is relatively small. [Rothe \(2010, 2012\)](#) generalize the method of [Firpo](#),

Fortin, and Lemieux (2009), for other recent developments, see, e.g., Inoue, Li, and Xu (2021), Sasaki, Ura, and Zhang (2022), Martinez-Iriarte, Montes-Rojas, and Sun (2022), Martinez-Iriarte and Sun (2022), and Martinez-Iriarte (2023). For a comprehensive survey on counterfactual distributions and decomposition methods, see Fortin, Lemieux, and Firpo (2011). Another branch of the literature related to this paper uses a combination of standard CQR with simulation exercises to evaluate distributional effects, such as UQPE. While it is feasible to calculate the unconditional distribution of an outcome variable using CQR (see, e.g., Autor, Katz, and Kearney (2005), Machado and Mata (2005), Melly (2005), and Chernozhukov, Fernández-Val, and Melly (2013)), this task is not obvious, at least compared to the ordinary least-squares (OLS) for the conditional mean. Since an analogue of the law of iterated expectations does not hold in the case of quantiles (de Castro, Costa, Galvao, and Zubelli (2023)), the CQR analysis cannot be directly employed to analyze unconditional quantiles (see the discussion in Fortin, Lemieux, and Firpo, 2011).

The remaining of the paper is organized as follows. Section 2 presents the main result that motivates the UQPE estimator based on CQR. Section 3 proposes an estimator for the UQPE and Section 4 derives its asymptotic properties. Section 5 studies the estimator’s finite sample performance using Monte Carlo experiments. Section 6 provides an empirical application. Section 7 concludes.

2 Quantile Partial Effects

In this section we introduce the unconditional quantile partial effect (UQPE), and the conditional quantile partial effect (CQPE). Some developments appear in Firpo, Fortin, and Lemieux (2009) and we reproduce them here for completeness. The relationship between them is the foundation for the UQPE estimator we discuss in the next section. In the following, the *unconditional* quantiles of Y are indexed by $\tau \in (0, 1)$, while the *conditional* quantiles Y given X are indexed by $\eta \in (0, 1)$.

2.1 UQPE in terms of CQPE

Consider a general model $Y = r(X, U)$, where $X = (X_1, X_2)'$. Here, Y is the dependent variable, X_1 is the target variable of interest and is a scalar, X_2 is a $(d - 1) \times 1$ vector consisting of other observable covariates, and U consists of unobservables. A leading example is the simple linear model $Y = \beta_0 + \beta_1 X_1 + X_2' \beta_2 + U$, such that the conditional η -quantile of Y given (X_1, X_2) is

$$Q_Y[\eta | X_1, X_2] = \beta_0(\tau) + \beta_1(\eta)X_1 + X_2' \beta_2(\eta). \quad (1)$$

The typical object of study of the standard conditional quantile regression (CQR) is the conditional quantile partial effect (CQPE) defined as

$$CQPE_{X_1}(\eta, x) := \frac{\partial Q_Y[\eta | X_1 = z, X_2 = x_2]}{\partial z} \Big|_{z=x_1}, \quad (2)$$

and corresponds to the marginal effect of X_1 on the conditional quantiles of the outcome when $X_1 = x_1$ and $X_2 = x_2$. In the case of model (1), $CQPE_{X_1}(\eta, x) = \beta_1(\eta)$ and estimation of this parameter follows from standard quantile regression methods.

To define an unconditional counterpart to $CQPE_{X_1}(\eta, x)$, we follow [Firpo, Fortin, and Lemieux \(2009\)](#). To that end, consider the counterfactual outcome

$$Y_{\delta, X_1} = r(X_1 + \delta, X_2, U),$$

where δ captures a small location change in the variable X_1 . Y_{δ, X_1} is the outcome we would observe if every individual receives an additional quantity δ of X_1 , while keeping X_2 , U , and the joint dependence between X_1 , X_2 , and U constant. Let $Q_Z[\tau]$ be the unconditional τ -quantile of the random variable Z . Then the unconditional quantile partial effect (UQPE) is defined as⁴

$$UQPE_{X_1}(\tau) := \lim_{\delta \rightarrow 0} \frac{Q_{Y_{\delta, X_1}}[\tau] - Q_Y[\tau]}{\delta}. \quad (3)$$

The $UQPE_{X_1}(\tau)$ is the marginal effect of a location shift in X_1 on the unconditional τ -quantile of the outcome.

The interpretation of $CQPE_{X_1}(\eta, x)$ and $UQPE_{X_1}(\tau)$ are different. The $CQPE_{X_1}(\eta, x)$ amounts to manipulating X_1 locally at x and evaluating a local impact on Y , i.e., it measures the effect of a marginal change in X_1 on the η -conditional quantile of Y . In other words, it only concerns to observations (i.e., say individuals) with $X = x$. The $UQPE_{X_1}(\tau)$ is obtained by what we may refer to as a global change in X_1 : everyone (not just those with $X = x$ covariates values) moves from X_1 to $X_1 + \delta$. Then, it looks at its associated impact on the τ -unconditional quantile of Y .⁵

[Firpo, Fortin, and Lemieux \(2009\)](#) show that under some mild conditions the following identification result holds:

$$UQPE_{X_1}(\tau) = -\frac{1}{f_Y(Q_Y[\tau])} \int \frac{\partial F_{Y|X}(Q_Y[\tau] | z, x_2)}{\partial z} \Big|_{z=x_1} dF_X(x), \quad (4)$$

⁴The UQPE can be defined for a vector of covariates as in [Firpo, Fortin, and Lemieux \(2009\)](#) resulting in a vector of UQPEs. [Martinez-Iriarte, Montes-Rojas, and Sun \(2022\)](#) provide an interpretation of the linear combination of UQPEs as a compensated counterfactual change.

⁵[Galvao and Wang \(2015\)](#) study quantile treatment effects with a continuous treatment. When treatment is continuous, taking values $t \in \mathcal{T}$, a subset of the real line, interest lies in the dose response function $t \mapsto Y(t)$, which is the potential outcome associated with different levels of the treatment. The quantile dose response functions aims at estimating the quantiles of $Y(t)$ for different values of t . This is different from here, where we estimates the effect on the quantiles of the *observed* outcome Y .

where $F_X(x)$ is short for the joint distribution, $F_{X_1, X_2}(x_1, x_2)$. In a similar manner, assuming differentiability of $F_{Y|X}(y|\cdot)$, from (2) we have that

$$CQPE_{X_1}(\eta, x) = -\frac{1}{f_{Y|X}(Q_Y[\eta|X=x]|x)} \frac{\partial F_{Y|X}(Q_Y[\eta|X=x]|z, x_2)}{\partial z} \Big|_{z=x_1}. \quad (5)$$

It is interesting to see that $CQPE_{X_1}(\eta, x)$ in equation (5) has a similar structure to $UQPE_{X_1}(\tau)$ in (4). Comparing the formulas in (4) and (5), one is able to see that even if the conditional quantile is equal to the corresponding unconditional, that is, $Q_Y[\tau|X=x] = Q_Y[\tau]$, one is *not* able to recover $UQPE_{X_1}(\tau)$ from $CQPE_{X_1}(\tau, \cdot)$ by simply integrating the latter over X . Moreover, it is usually the case that $Q_Y[\tau|X=x] \neq Q_Y[\tau]$. Thus, first we need to match conditional and unconditional quantiles and then re weight them appropriately to recover $UQPE_{X_1}(\tau)$ from $CQPE_{X_1}(\cdot, \cdot)$.

The following matching map, introduced by [Firpo, Fortin, and Lemieux \(2009, p.959\)](#), is an important tool to relate the CQPE and UQPE:

$$\xi_\tau(x) = \{\eta : Q_Y[\eta|X=x] = Q_Y[\tau]\}. \quad (6)$$

The map $\xi_\tau(x) : (0, 1) \times \mathbb{R}^d \mapsto (0, 1)$ corresponds to the quantile index(es) in the conditional model, η , that produces the closest match with the unconditional quantiles τ for different values of x . In Section 2.3 we analyze this map in detail. For now, we assume that $\xi_\tau(x)$ is a singleton. Therefore, we have that, for every x , $Q_Y[\xi_\tau(x)|X=x] = Q_Y[\tau]$. Under this condition, it is simple to formalize the relationship between CQPE and UQPE. Note that the $CQPE_{X_1}$ in equation (5) evaluated at $\eta = \xi_\tau(x)$ can be written as

$$CQPE_{X_1}(\xi_\tau(x), x) = -\frac{1}{f_{Y|X}(Q_Y[\tau]|x)} \frac{\partial F_{Y|X}(Q_Y[\tau]|z, x_2)}{\partial z} \Big|_{z=x_1}.$$

Now, rearranging $CQPE_{X_1}(\xi_\tau(x), x)$ above and substituting into equation (4) yields

$$UQPE_{X_1}(\tau) = \int CQPE_{X_1}(\xi_\tau(x), x) \frac{f_{Y|X}(Q_Y[\tau]|x)}{f_Y(Q_Y[\tau])} dF_X(x). \quad (7)$$

The weights in (7) can be rearranged as

$$\frac{f_{Y|X}(Q_Y[\tau]|x)}{f_Y(Q_Y[\tau])} f_X(x) = \frac{f_{Y,X}(Q_Y[\tau], x)}{f_Y(Q_Y[\tau]) f_X(x)} f_X(x) = f_{X|Y}(x|Q_Y[\tau]).$$

Finally, equation (7) becomes a reverse projection as

$$UQPE_{X_1}(\tau) = E[CQPE_{X_1}(\xi_\tau(X), X)|Y = Q_Y[\tau]]. \quad (8)$$

Equation (8) shows that the UQPE is in fact a *local* weighted average of CQPE effects “near”

the unconditional τ -quantile of Y . As noted above, the τ -th unconditional quantile of interest may be different from the (random) $\xi_\tau(X)$ -th conditional quantiles used inside the integral. The preceding informal discussion is summarized in the lemma.

Lemma 1. *Let the following assumptions hold: (i) $UQPE_{X_1}(\tau)$ is identified by (4); (ii) the matching function defined in (6) is a singleton; (iii) $F_{Y|X}(y|x)$ and $Q_Y[\tau|X = x]$ are differentiable with respect to x_1 ; (iv) $f_{Y|X}$, f_Y and f_X are strictly positive. Then (8) holds.*

As mentioned above, sufficient conditions for (i) are laid out in [Firpo, Fortin, and Lemieux \(2009\)](#). Regarding (ii), see Assumption 1 stated below for sufficient conditions for $\xi_\tau(x)$ to be singleton. The rest of the assumptions are customary regularity conditions.⁶

2.2 Description of the procedure

Given the discussion above, it is possible to compute the UQPE from CQPE using equation (8). Consider a linear conditional quantile model as in (1), then $CQPE_{X_1}(\xi_\tau(x), x) = \beta_1(\xi_\tau(x))$. This implies that (8) becomes a weighted average of matched slopes. Figures 1 and 2 illustrate how the procedure works in two different linear cases. The figures plot both the unconditional quantile, $Q_Y[\tau]$ (red line) and conditional quantiles, $Q_{Y|X}[\eta]$ (blue lines), as well as the conditional density $f_{X|Y}$ (green curve). An informal description is the following:

1. Identify the unconditional τ quantile, $Q_Y[\tau]$, say $Q_Y[0.50]$ as illustrated in the figures for the unconditional median, and drawn in a horizontal (red) line.
2. Notice that for each η , the conditional quantiles $Q_{Y|X}[\eta]$ (blue lines) intersect the unconditional quantile $Q_Y[\tau]$ (horizontal red line); in the figures, we illustrate this for $\{x_1, x_2, x_3, x_4\}$ values of X that correspond to $\eta = \xi_\tau(x)$ values $\{0.80, 0.60, 0.40, 0.20\}$ respectively.
3. The UQPE is the weighted average – with weights given by density $f_{X|Y}(x|Q_Y[\tau])$ (green curve) – of the intersected slopes on the conditional quantile models.

The figures are useful for analyzing the source of the variation in the UQPE across different unconditional quantiles. For example, in Figure 1, the CQPE slopes are the same across conditional quantiles η , which implies that the CQPE is constant across quantiles. Even if the weights change with τ , this is irrelevant, because the slopes are constant.

On the other hand, in Figure 2, the CQPE slopes exhibit some variation across conditional quantiles η . This heterogeneity can be present even if the weights are not a function of τ . The UQPE is then constructed as a weighted average of those. An additional source of variation is given by the potential different conditional densities of X given $Y = Q_Y[\tau]$. The UQPE will then be based on the different CQPE and the corresponding density weights. Section 3 below formalizes the estimator.

⁶[Rothe \(2010\)](#) extends the identification of UQPE to nonseparable triangular models with endogenous regressors via a control variable approach.

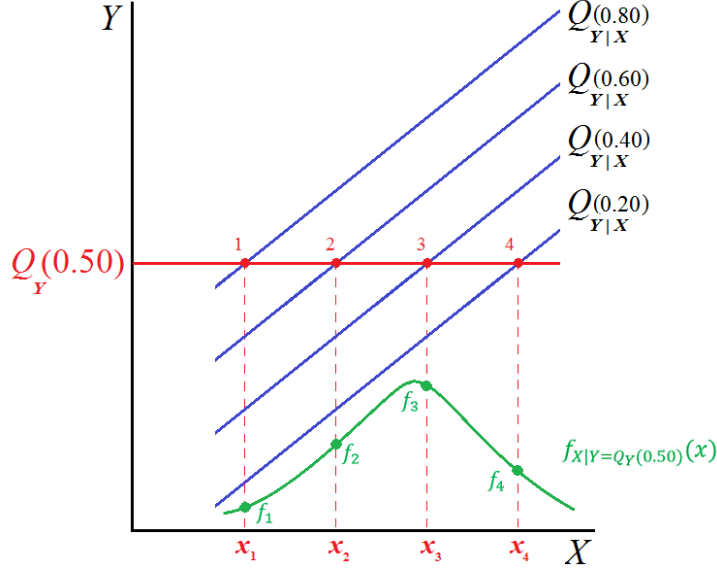


Figure 1: Constant CQPE.

2.3 The matching map ζ_τ

Now we study the matching map. Equation (6) defined the map as

$$\zeta_\tau(x) = \{\eta : Q_Y[\eta|X = x] = Q_Y[\tau]\}.$$

For a fixed covariate value $X = x$, the map $\tau \mapsto \zeta_\tau(x)$ describes how the unconditional distribution maps on the conditional one. In general, $\zeta_\tau(x)$ may vary across the value of covariates as well. Note that it is entirely possible that $\tau \neq \zeta_\tau(x)$.

For the purpose of this paper it is important that $\zeta_\tau(x)$ is unique. But, generally, three situations may occur. First, $\zeta_\tau(x)$ is unique when $F_{Y|X}(y|x)$ is strictly increasing. In this case, there can be at most one η that satisfies equation (6). To see this, note that $Q_Y[\eta|X = x] = Q_Y[\tau]$ is identical to $\eta = F_{Y|X}(Q_Y[\tau]|x)$, so that $\zeta_\tau(x)$ is unique. Second, $\zeta_\tau(x)$ might be an interval. For example if $F_{Y|X}(y|x)$ has a jump discontinuity at $y = Q_Y[\tau]$, but it is otherwise continuous and strictly increasing, then $\zeta_\tau(x) = [\lim_{y \uparrow Q_Y[\tau]} F_{Y|X}(y|x), F_{Y|X}(Q_Y[\tau]|x)]$. See Figure 3a below. Third, $\zeta_\tau(x)$ might be empty for some x . For example, suppose that $F_{Y|X}(y|x)$ is continuous, and aside from a flat interval, it is strictly increasing. If $Q_Y[\tau]$ is in the interior of the interval mapping to the flat interval, then we cannot have $Q_Y[\eta|X = x] = Q_Y[\tau]$. This is illustrated in Figure 3b below.

Now we provide an example to illustrate the how to explicitly write matching function $\zeta_\tau(x)$ using a linear QR model.

Example 1. Consider the model $Y = \alpha_0 + \alpha_1 X_1 + (1 + \theta X_1)U$ with $X \perp U$. By standard computations, if $1 + \theta x_1 > 0$, then $Q_Y[\eta|X_1 = x_1] = \alpha_0 + \alpha_1 x_1 + (1 + \theta x_1)Q_U[\eta]$. To find $\zeta_\tau(x_1)$ we need the level η

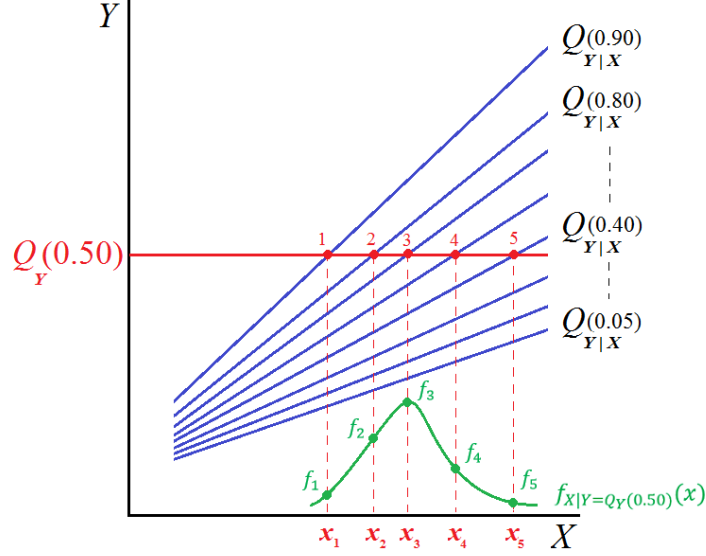


Figure 2: Heterogeneous CQPE.

such that $Q_Y[\eta|X_1 = x_1] = Q_Y[\tau]$. Thus,

$$\xi_\tau(x_1) = F_U \left(\frac{Q_Y[\tau] - \alpha_0 - \alpha_1 x_1}{1 + \theta x_1} \right). \quad (9)$$

Remark 1. If the matching function is the identity function: $\xi_\tau(X) = \{\eta : Q_Y[\eta|X = x] = Q_Y[\tau]\} = \tau$, Then, by equation (8), $UQPE_{X_1}(\tau)$ can be written as

$$UQPE_{X_1}(\tau) = \int CQPE_{X_1}(\eta, x) \frac{f_{X|Y}(x|Q_Y[\tau])}{f_X(x)} f_X(x) dx,$$

which is the parameter of interest of Lee (2021): a weighted average quantile derivative. Here the weight is $\frac{f_{X|Y}(x|Q_Y[\tau])}{f_X(x)}$. If the matching function is not the identity, then our parameter is not covered by the methods of Lee (2021).

Since the estimation of the matching function is crucial for our results, we carry out many Monte Carlo simulations to assess the quality of the estimator proposed below in (14). In both the Monte Carlo simulations and the empirical application we plot the estimated matching function for different quantiles against the values of the of X_1 . The shape of the matching function can give an indication of the effect of X_1 on Y , as Example 1 shows.

2.4 η -heterogeneity vs. τ -heterogeneity

Here we study how does heterogeneity in the conditional effects across η -quantiles propagate to heterogeneity in unconditional effects across τ -quantiles. We refer to the latter as τ -heterogeneity,

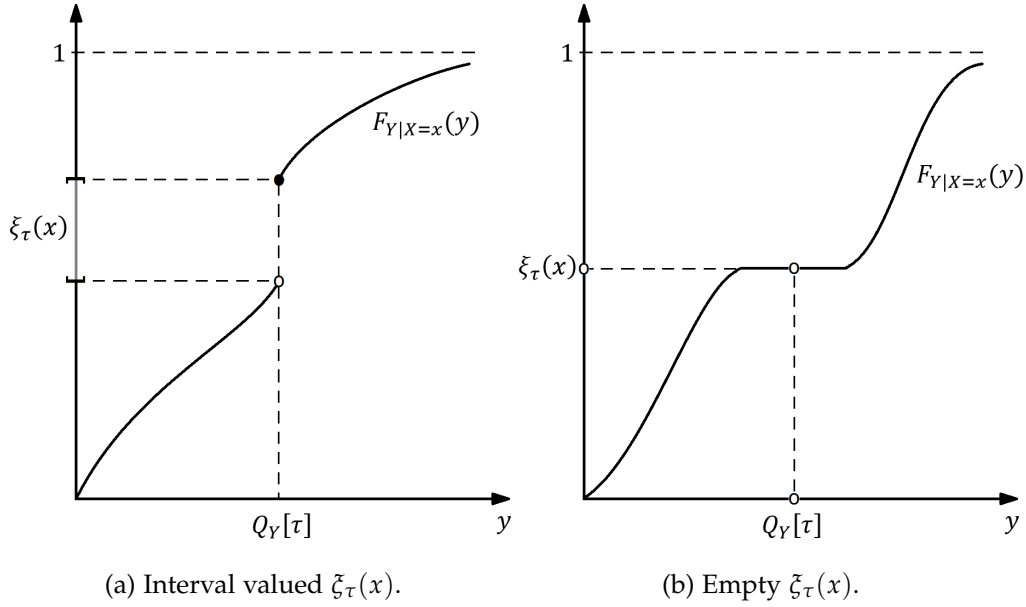


Figure 3: Non-uniqueness of $\xi_\tau(x)$.

and to the former as η -heterogeneity. More precisely:

$$\eta\text{-heterogeneity} := \frac{\partial CQPE_{X_1}(\eta, x)}{\partial \eta},$$

and

$$\tau\text{-heterogeneity} := \frac{dUQPE_{X_1}(\tau)}{d\tau}.$$

While the η -heterogeneity depends on (a fixed) x , we keep this dependence implicit. Using the chain rule, we obtain the following marginal change in the unconditional quantile effect with respect to τ :

$$\begin{aligned} \frac{dUQPE_{X_1}(\tau)}{d\tau} &= \int_x \frac{\partial CQPE_{X_1}(\eta, x)}{\partial \eta} \Big|_{\eta=\xi_\tau(x)} \frac{\partial \xi_\tau(x)}{\partial \tau} f_{X|Y}(x|Q_Y[\tau]) dx \\ &\quad + \frac{dQ_Y[\tau]}{d\tau} \int_x CQPE_{X_1}(\xi_\tau(x), x) \frac{df_{X|Y}(x|y)}{dy} \Big|_{y=Q_Y[\tau]} dx. \end{aligned}$$

The first term averages across the η -heterogeneity. In general, even if there is no η -heterogeneity, we may still have non-zero τ -heterogeneity through the second term. An exception is when $CQPE_{X_1}(\eta, x) = \beta_1$, since in this case, $UQPE_{X_1}(\tau) = \beta_1$ as given in equation (8). This is the case of Example 1 above, which we discuss in more details now.

Example 1 (Continued). Here, $CQPE_{X_1}(\eta, x) = \beta_1(\eta) = \alpha_1 + \theta Q_U[\eta]$. The η -heterogeneity is gov-

etermined by the parameter θ :

$$\frac{d\beta_1(\eta)}{d\eta} = \theta \frac{dQ_U[\eta]}{d\eta}.$$

The unconditional effect is $UQPE_{X_1}(\tau) = E[\beta_1(\xi_\tau(X_1))|Y = Q_Y[\tau]]$. It can be shown that

$$\frac{dUQPE_{X_1}(\tau)}{d\tau} = \frac{dQ_Y[\tau]}{d\tau} \int_{x_1} \frac{\theta}{1 + \theta x_1} f_{X_1|Y}(x_1|Q_Y[\tau]) dx_1 + \frac{dQ_Y[\tau]}{d\tau} \int_{x_1} \beta_1(\xi_\tau(x_1)) \frac{df_{X_1|Y}(x_1|y)}{dy} \Big|_{y=Q_Y[\tau]} dx_1.$$

Now, if $\theta = 0$, there is no η -heterogeneity, so the first term in the above expression for the τ -heterogeneity is 0. The second term is also 0, since $\beta_1(\eta) = \alpha_1 = \beta_1(\xi_\tau(x_1))$:

$$\begin{aligned} \int_{x_1} \beta_1(\xi_\tau(x_1)) \frac{df_{X_1|Y}(x_1|y)}{dy} \Big|_{y=Q_Y[\tau]} dx_1 &= \frac{dQ_Y[\tau]}{d\tau} \alpha_1 \int_{x_1} \frac{df_{X_1|Y}(x_1|y)}{dy} \Big|_{y=Q_Y[\tau]} dx_1 \\ &= \frac{dQ_Y[\tau]}{d\tau} \alpha_1 \underbrace{\frac{d}{dy} \int_{x_1} f_{X_1|Y}(x_1|y) dx_1}_{=1} \Big|_{y=Q_Y[\tau]} \\ &= 0, \end{aligned}$$

provided we can interchange derivatives with integration. Thus, no η -heterogeneity implies no τ -heterogeneity.

3 Estimator

In this section we describe a two-step estimator of $UQPE_{X_1}(\tau)$, which is based on the reverse projection in equation (8). The asymptotic properties are discussed in later sections.

Assume first that

$$Q_Y[\eta|X_1 = x_1, X_2 = x_2] = x_1\beta_1(\eta) + x_2'\beta_2(\eta) = x'\beta(\eta), \quad (10)$$

where $\beta = (\beta_1, \beta_2)'$. Note that x_2 has to include a constant for correct specification. In this paper, we use the conditional quantile function in (10) to estimate the UQPE. Using this quantile regression model has advantages. First, it allows the researcher to directly model the outcome variable Y as a function of observable covariates X , instead of modeling the recentered influence function. This is important because it may be simpler to relate the variable of interest directly from the economic theory or existing literature, than modeling the influence function. Second, practical estimation of (10) is simple, as we discuss below.

Under (10), $CQPE_{X_1}(\xi_\tau(x), x) = \beta_1(\xi_\tau(x))$. Equation (8) then has the convenient form

$$UQPE_{X_1}(\tau) = E[\beta_1(\xi_\tau(X))|Y = Q_Y[\tau]]. \quad (11)$$

Our proposed estimator is a nonparametric regression of $\{\beta_1(\xi_\tau(x_i))\}_{i=1}^n$ on $\{y_i\}_{i=1}^n$ evaluated at

$Q_Y[\tau]$. To implement this method in practice we are required to estimate $\beta_1(\xi_\tau(x))$ and $Q_Y[\tau]$.

To estimate $\beta_1(\xi_\tau(x_i))$ we first use CQR methods, and estimate $\beta(\eta)$ for a grid of m values of η 's given by $\mathcal{H}_m = \{\epsilon < \eta_1 < \dots < \eta_j < \dots < \eta_m < 1 - \epsilon\}$, $\epsilon \in (0, \frac{1}{2})$. In the standard linear case we have that for a given value of η_j , and a sample $\{y_i, x_i\}_{i=1}^n$, we simply apply standard quantile regression methods as

$$(\hat{\beta}_1(\eta_j), \hat{\beta}_2(\eta_j)')' = \hat{\beta}(\eta_j) = \arg \min_b \frac{1}{n} \sum_{i=1}^n \rho_{\eta_j}(y_i - x_i' b), \quad (12)$$

where $\rho_\tau(u) = u(\tau - 1[u < 0])$ is the [Koenker and Bassett \(1978\)](#) check function. We also estimate the unconditional quantile $Q_Y[\tau]$ by

$$\hat{Q}_Y[\tau] = \arg \min_q \frac{1}{n} \sum_{i=1}^n \rho_\tau(y_i - q). \quad (13)$$

To find the matched coefficients $\hat{\beta}_1(\hat{\xi}_\tau(x_i))$, we employ the two previous estimates as following. Let

$$\hat{\xi}_\tau(x_i) = \begin{cases} \eta_1 \in \mathcal{H}_m & \text{if } \hat{Q}_Y[\tau] < x_i' \hat{\beta}(\eta_1); \\ \eta_j \in \mathcal{H}_m & \text{if } \{x_i' \hat{\beta}(\eta_{j-1}) \leq \hat{Q}_Y[\tau] < x_i' \hat{\beta}(\eta_j)\} \text{ for } j = 2, \dots, m-1; \\ \eta_m \in \mathcal{H}_m & \text{if } \hat{Q}_Y[\tau] \geq x_i' \hat{\beta}(\eta_m), \end{cases} \quad (14)$$

for $i = 1, \dots, n$.⁷ Finally, to estimate the $UQPE_{X_1}(\tau)$, we use a Nadaraya-Watson type-estimator, using the preliminary estimators:

$$\hat{E}[\hat{\beta}_1(\hat{\xi}_\tau(X)) | Y = \hat{Q}_Y[\tau]] = \frac{\sum_{i=1}^n K_h(y_i - \hat{Q}_Y[\tau]) \cdot \hat{\beta}_1(\hat{\xi}_\tau(x_i))}{\sum_{i=1}^n K_h(y_i - \hat{Q}_Y[\tau])}, \quad (15)$$

where K_h is the rescaled kernel $K_h(u) := \frac{1}{h} K(\frac{u}{h})$. The estimator in (15) avoids the curse of dimensionality because it is a nonparametric regression on just one regressor: Y . Indeed, the dimension of X enters in the CQR estimation and in the matching function.

Equation (15) highlights the main benefit of our proposed approach: obtaining the unconditional effect is an easy follow-up from the conditional effects. If the researcher, as is usually the case, has estimated a grid of CQR coefficients, then, after they are ‘‘matched’’ according to (14), they can be averaged following (15) to yield the unconditional effect for the desired quantile level. Moreover, notice that the second nonparametric step in equation (15) does not suffer from the curse of dimensionality, in the sense that it is a reverse regression where the one-dimensional

⁷The above estimator relies on monotonicity of CQR such that there is only one match. In practice, this needs to be checked in small samples as multiple crossings may occur if x_i is very different from \bar{x} . Then an algorithm could be implemented such as taking the average of the selected β_1 or a rearrangement of estimated quantiles (see, for instance, [Chernozhukov, Fernández-Val, and Galichon \(2010\)](#) for a discussion about quantile crossings). Furthermore, if the unconditional quantile does not lie inside the intervals generated by the grid of conditional quantile values, then we impute either the minimum if it is below, or the maximum if it is above.

outcome is the regressor.⁸

Remark 2. An alternative approach to estimating $UQPE_{X_1}(\tau)$ based on (11) is a linear regression of $\hat{\beta}_1(\hat{\xi}_\tau(X))$ on a constant and Y . The predicted fit at $Y = \hat{Q}_Y[\tau]$ is an easy-to-compute approximation to $UQPE_{X_1}(\tau)$. Yet another option is to do a local linear regression. This estimator may help reduce the bias in lower and higher quantiles. The estimator is $\hat{a}_{\tau,0} + \hat{a}_{\tau,1}\hat{Q}_Y[\tau]$, where $(\hat{a}_{\tau,0}, \hat{a}_{\tau,1})'$ solve

$$(\hat{a}_{\tau,0}, \hat{a}_{\tau,1})' = \arg \min_{a_{\tau,0}, a_{\tau,1}} \sum_{i=1}^n K_h(y_i - \hat{Q}_Y[\tau]) \left[\beta_1(\hat{\xi}_\tau(x_i)) - a_{\tau,0} - a_{\tau,1} \left(\frac{y_i - \hat{Q}_Y[\tau]}{h} \right) \right]^2.$$

A study of the properties of this estimator in this particular setting is left for future research.

Next we provide a concise algorithm to compute the UQPE estimator for a given τ .

Algorithm 1 UQPE Computation

Data: $\{y_i, x_i\}_{i=1}^n$ with y_i a scalar and $x_i \in \mathbb{R}^d$ for $d \geq 1$.

Input: $\tau \in (0, 1)$, $\epsilon \in (0, 1/2)$, $m \in \mathbb{N}$, bandwidth h , and kernel function K_h .

1. Construct the grid

$$\mathcal{H}_m = \{\epsilon < \eta_1 < \dots < \eta_j < \dots < \eta_m < 1 - \epsilon\}.$$

2. Estimate the QR coefficients $\hat{\beta}(\eta_j)$ for $j = 1, 2, \dots, m$ as in (12) and $\hat{Q}_Y[\tau]$ as in (13).
 3. For each observation $i = 1, 2, \dots, n$, construct $x'_i \hat{\beta}(\eta_j)$ for $j = 1, 2, \dots, m$.
 4. For each observation $i = 1, 2, \dots, n$, compute the matching function $\hat{\xi}_\tau(x_i)$ as in (14) and then compute $\hat{\beta}(\hat{\xi}_\tau(x_i))$.
 5. For each observation $i = 1, 2, \dots, n$ compute the kernel function $K_h(y_i - \hat{Q}_Y[\tau])$ and estimate (15) to compute the UQPE.
-

Note that this procedure is based on the initial estimate of a QR process for $\eta \in \mathcal{H}_m$, a typical output in QR analysis, and the unconditional quantile of Y . Then, for each observation compute the estimated conditional quantile functions $x'_i \hat{\beta}_\tau(\eta)$ for $\eta \in \mathcal{H}_m$ (point 3). The matching function involves a simple binary argument from the results in point 3. Then the algorithm needs to retrieve the corresponding QR coefficient (already estimated in point 2) delivered by the match. Finally compute a weighted average using the kernels as weights (point 5). Importantly, if several values of τ are to be computed, then points 1, 2 and 3 do not have to be redone.

⁸The RIF-nonparametric approach is potentially subject to the curse of dimensionality since multiple regressors may enter the nonparametric second step.

4 Asymptotic Theory

This section derives the asymptotic properties of the two-step estimator. First, we study the first step, and establish an asymptotic linear representation and rate of convergence for the conditional quantile regression coefficients as a function of the matched quantiles. Second, we study the asymptotic properties of the nonparametric regression in the second step.

4.1 Structural QR and Matched Quantiles

The following assumptions are needed to establish that $\hat{\beta}_1(\hat{\xi}_\tau(x)) - \beta_1(\xi_\tau(x)) = O_p(n^{-1/2})$, where $\hat{\beta}_1(\hat{\xi}_\tau(x))$ is computed according to (14).

Assumption 1. Let $\{y_i, x_i\}_{i=1}^n$ be a random sample of independent and identically distributed (iid) observations with y_i a scalar and $x_i \in \mathbb{R}^d$ that satisfy the following properties:

1. The conditional quantiles are linear: $Q_Y[\eta|X = x] = x'\beta(\eta)$, $\eta \in [\epsilon, 1 - \epsilon]$, $\epsilon \in (0, \frac{1}{2})$, with $X \in \mathbb{R}^d$ and $E|X| < \infty$.
2. For every x in the support of X , $f_{Y|X}(y|x)$ is bounded away from zero.
3. The conditional quantile regression estimators satisfy

$$\begin{aligned} \hat{\beta}(\eta) - \beta(\eta) &= E [f_{Y|X}(X'\beta(\eta)|X)XX']^{-1} \frac{1}{n} \sum_{i=1}^n (\eta - \mathbb{1}\{y_i \leq x_i'\beta(\eta)\}) x_i + o_p(n^{-1/2}) \\ &= \frac{1}{n} \sum_{i=1}^n \Psi_i(\eta) + o_p(n^{-1/2}), \end{aligned}$$

uniformly in $\eta \in [\epsilon, 1 - \epsilon]$, $\epsilon \in (0, \frac{1}{2})$, and $\eta \mapsto E [f_{Y|X}(X'\beta(\eta)|X)XX']$ has uniformly bounded derivatives.

4. The unconditional quantile estimator satisfies

$$\begin{aligned} \hat{Q}_Y[\tau] - Q_Y[\tau] &= f_Y(Q_Y[\tau])^{-1} \frac{1}{n} \sum_{i=1}^n (\tau - \mathbb{1}\{y_i \leq Q_Y[\tau]\}) \\ &= \frac{1}{n} \sum_{i=1}^n \psi_i(\tau) + o_p(n^{-1/2}). \end{aligned}$$

5. The grid of quantiles $\{\epsilon < \eta_1 < \dots < \eta_j < \dots < \eta_m < 1 - \epsilon\}$, $\epsilon \in (0, \frac{1}{2})$, satisfies $\Delta\eta = o(n^{-1/2})$ as $n \rightarrow \infty$ for $\Delta\eta := \eta_j - \eta_{j-1}$, $j = 2, \dots, m$, and $\eta_1 = \epsilon$ and $\eta_m = 1 - \epsilon$ for a small $\epsilon > 0$.

The conditions in Assumption 1 are very mild. Assumption 1.1 imposes linearity of the model, and condition 1.2 is very standard in the QR literature, see, e.g., [Koenker \(2005\)](#). Assumptions 1.1 and 1.2, allow us to write $F_{Y|X}(x'\beta(\eta)|x) = \eta$, so that $x'\dot{\beta}(\eta) = f_{Y|X}(x'\beta(\eta)|x)^{-1} > 0$,

where $\dot{\beta}(\eta)$ is the Jacobian vector: the derivative of the map $\eta \mapsto \beta(\eta)$. This quantity appears in the denominator of the influence function of $\hat{\xi}_\tau(x)$. Moreover, Assumption 1.2 states that $f_{Y|X}(y|x)$ is bounded away from zero. This in turn implies that $y \mapsto F_{Y|X}(y|x)$ is strictly increasing. Assumption 1.3 is a uniform Bahadur representation for the QR estimator. It is established in Lemma 3 in Ota, Kato, and Hara (2019). See also Theorem 3 in Angrist, Chernozhukov, and Fernández-Val (2006). It implies $\sup_{\eta \in [\epsilon, 1-\epsilon]} |\hat{\beta}(\eta) - \beta(\eta)| = O_p(n^{-1/2})$ and the stochastic equicontinuity of the process $\tau \mapsto \sqrt{n}(\hat{\beta}(\eta) - \beta(\eta))$ on $[\epsilon, 1-\epsilon]$. Condition 1.4 is a simple linear representation for the unconditional quantile. Sufficient conditions for Assumption 1.4 are given in Serfling (1980). Finally, Assumption 1.5 requires that the grid for the matching function becomes denser as the sample size increases. This condition has appeared in the QR literature. Chernozhukov, Fernández-Val, and Melly (2013, Remark 3.1 p.2220) provide a similar condition when computing counterfactual distributions.

The next result provides a rate of convergence and a linear representation for $\hat{\beta}_1(\hat{\xi}_\tau(x)) - \beta_1(\xi_\tau(x)) = O_p(n^{-1/2})$.

Theorem 1. *Under Assumption 1, the CQR coefficient of X_1 evaluated at the random quantile $\hat{\xi}_\tau(x)$ satisfies $\hat{\beta}_1(\hat{\xi}_\tau(x)) - \beta_1(\xi_\tau(x)) = O_p(n^{-1/2})$ and can be represented as*

$$\hat{\beta}_1(\hat{\xi}_\tau(x)) - \beta_1(\xi_\tau(x)) = \hat{\beta}_1(\xi_\tau(x)) - \beta_1(\xi_\tau(x)) + \dot{\beta}_1(\xi_\tau(x))(\hat{\xi}_\tau(x) - \xi_\tau(x)) + o_p(n^{-1/2}),$$

where

$$\hat{\xi}_\tau(x) - \xi_\tau(x) = -\frac{1}{x' \dot{\beta}(\xi_\tau(x))} \frac{1}{n} \sum_{i=1}^n x' \Psi_i(\xi_\tau(x)) + \frac{1}{x' \dot{\beta}(\xi_\tau(x))} \frac{1}{n} \sum_{i=1}^n \psi_i(\tau) + o_p(n^{-1/2}).$$

Here, $\dot{\beta}_1(\xi_\tau(x))$ is the β_1 component of the Jacobian vector $\dot{\beta}(\xi_\tau(x))$.

4.2 Nadaraya-Watson Estimator

Our parameter of interest given in (11) is

$$UQPE_{X_1}(\tau) = E[\beta_1(\xi_\tau(X)) | Y = Q_Y[\tau]],$$

and we propose the following nonparametric regression Nadaraya-Watson-type estimator:

$$\widehat{UQPE}_{X_1}(\tau) = \hat{E}[\hat{\beta}_1(\hat{\xi}_\tau(X)) | Y = \hat{Q}_Y[\tau]] = \frac{\sum_{i=1}^n K_h(y_i - \hat{Q}_Y[\tau]) \cdot \hat{\beta}_1(\hat{\xi}_\tau(x_i))}{\sum_{i=1}^n K_h(y_i - \hat{Q}_Y[\tau])}.$$

The unfeasible (oracle) version is denoted by

$$\widetilde{UQPE}_{X_1}(\tau) = \hat{E}[\beta_1(\xi_\tau(X)) | Y = Q_Y[\tau]] = \frac{\sum_{i=1}^n K_h(y_i - Q_Y[\tau]) \cdot \beta_1(\xi_\tau(x_i))}{\sum_{i=1}^n K_h(y_i - Q_Y[\tau])}.$$

We maintain the following assumptions.

Assumption 2. $K(u)$ is a symmetric function around 0 that satisfies: (i) $\int K(u)du = 1$; (ii) For $r > 2$, $\int u^j K(u)du = 0$ when $j = 1, \dots, r-1$, and $\int |u|^r K(u)du < \infty$; (iii) $\int K'(u)du = 0$; (iv) $u^j K(u) \rightarrow 0$ as $u \rightarrow \pm\infty$ for $j=1, \dots, r+1$; (v) $\sup_{u \in \mathbb{R}} |K''(u)| < \infty$; (vi) $\int K'(u)^2 du < \infty$ and $\int uK'(u)^2 du < \infty$.

This assumption requires we use a r -th order kernel with $r > 2$. This is to remove the bias introduced by evaluating the kernel estimator at the estimated quantile. A popular one is the 4-th order Gaussian kernel (see Section 2.7.2 in Pagan and Ullah (1999)): $K(u) = \frac{3-u^2}{2}\phi(u)$ where $\phi(u)$ is the pdf of a standard normal. This kernel satisfies Assumption 2.⁹

Assumption 3. (i) The density of Y is $r+1$ times continuously differentiable, with uniformly bounded derivatives; (ii) The joint density $f_{Y,X}(y, x)$ is $r+1$ times continuously differentiable, with uniformly bounded derivatives for every x in the support of X .

Assumption 4. As $n \rightarrow \infty$, the bandwidth satisfies: (i) $h \rightarrow 0$; (ii) $(nh)^{1/2}h^r \rightarrow 0$, (iii) $nh^5 \rightarrow \infty$.

In order for a Assumption 4 to hold, we need that $1 + 2r > 5$, which implies $r > 2$. This is in line with the requirement of $r > 2$ in Assumption 2. For example, for $r = 4$, if $h \propto n^{-1/6}$, then Assumption 4 holds.

Assumption 5. The following approximation rate holds for $\hat{\xi}_\tau$: $E \left[(n^{1/4} [\beta_1(e(X)) - \beta_1(\xi_\tau(X))])^2 \right] \Big|_{e=\hat{\xi}_\tau} = o_p(1)$.

Theorem 2. Let Assumptions 1, 2, 3, 4, and 5 hold. Then, as $n \rightarrow \infty$,

$$\widehat{UQPE}_{X_1}(\tau) = \widetilde{UQPE}_{X_1}(\tau) + o_p(n^{-1/2}h^{-1/2}).$$

This theorem states that the preliminary estimators of the CQR slopes, the matched quantiles and the unconditional quantile of Y vanish asymptotically because they converge at a faster rate: $n^{-1/2}$ as opposed to $n^{-1/2}h^{-1/2}$. Moreover, the asymptotic distribution of the unfeasible estimator $\widetilde{UQPE}_{X_1}(\tau)$ is well-known and can be readily established.

The following assumption is customary in order to apply the Lindeberg-Feller Central Limit Theorem.

Assumption 6. (i) For $U_\tau := \beta_1(\xi_\tau(X)) - E[\beta_1(\xi_\tau(X))|Y]$, and $\gamma > 0$, $E[|U_\tau|^{2+\gamma}|Y] < C < \infty$ a.s. for some C ; (ii) $\int |K(u)|^{2+\gamma} du < \infty$; (iii) The map $y \mapsto E[\beta_1(\xi_\tau(X))|Y = y]$ is $r+1$ times continuously differentiable, with uniformly bounded derivatives; (iv) The map $y \mapsto \sigma_\tau^2(y) := E[U_\tau^2|Y = y]$ is continuous.

⁹Recall that the odd moments of a standard normal random variable are 0, and the first three even moments are 1, 3, and 15. We have $\int_{\mathbb{R}} K(u)du = 1$, $\int_{\mathbb{R}} uK(u)du = 0$, $\int_{\mathbb{R}} u^2K(u)du = 0$, $\int_{\mathbb{R}} u^3K(u)du = 0$, and $\int_{\mathbb{R}} u^4K(u)du = -3$. The first derivative of the kernel, using the recursive fact that $\phi'(u) = -u\phi(u)$, is $K'(u) = -\frac{5}{2}u\phi(u) + \frac{1}{2}u^3\phi(u)$, so that $\int_{\mathbb{R}} K'(u)du = 0$. Moreover, $u^j K(u) \rightarrow 0$ as $u \rightarrow \pm\infty$ for all j since the exponential rate of $\phi(u)$ decreases faster than the polynomial rate. By integration by parts, and using $u^j K(u) \rightarrow 0$ as $u \rightarrow \pm\infty$, we have that $\int u^j K'(u)du = -\int ju^{j-1}K(u)du$ which is 0 for $j = 2, 3, 4$ and $\int u^5 K'(u)du = -\int 5u^4 K(u)du = 15$. Finally, the second derivative is given by $K''(u) = -\frac{5}{2}\phi(u) + 4u^2\phi(u) - \frac{1}{2}u^4\phi(u)$ which is bounded since $u^j K(u) \rightarrow 0$ as $u \rightarrow \pm\infty$. Finally, using $K'(u) = -\frac{5}{2}u\phi(u) + \frac{1}{2}u^3\phi(u)$ it is easily verified that $\int K'(u)^2 du < \infty$ and $\int K'(u)^2 u du < \infty$.

Corollary 1. Let Assumptions 1, 2, 3, 4, 5 and 6 hold. Then, as $n \rightarrow \infty$,

$$\sqrt{nh} \left(\widehat{UQPE}_{X_1}(\tau) - UQPE_{X_1}(\tau) \right) \xrightarrow{d} N \left(0, \sigma_\tau^2(Q_Y[\tau]) f_Y(Q_Y[\tau])^{-1} \int K(u)^2 du \right).$$

Remark 3. The practical computation of the asymptotic variance-covariance matrix in Corollary 1 is difficult due to the presence of preliminary estimators in the nonparametric regression. Thus, in practice, we employ resampling approach for inference. There is an extensive literature on constructing nonparametric confidence bands for functions. We refer the reader to [Härdle and Bowman \(1988\)](#) and [Hall and Horowitz \(2013\)](#) and references therein for resampling methods.

We describe now the implementation of the pairwise bootstrap procedure.

1. Estimate $\{\hat{\beta}(\eta)\}$ for a given grid $\mathcal{H}_m = \{\eta_1, \dots, \eta_m\}$ and $\hat{Q}_Y[\tau]$, then compute $\widehat{UQPE}_{X_1}(\tau)$ using the sample $\{y_i, x_i\}_{i=1}^n$.
2. Compute samples with replacement $\{y_i^{*b}, x_i^{*b}\}_{i=1}^n$, for $b = 1, \dots, B$, and estimators $\{\hat{\beta}^{*b}(\eta)\}$ for \mathcal{H}_m , $\hat{Q}_Y^{*b}[\tau]$ and $\widehat{UQPE}_{X_1}^{*b}(\tau)$.
3. Compute the standard deviation from the bootstrap sample,

$$\hat{\sigma}_{UQPE}^* = \sqrt{\frac{1}{B} \sum_{b=1}^B \left(\widehat{UQPE}_{X_1}^{*b}(\tau) - \overline{\widehat{UQPE}_{X_1}^*}(\tau) \right)^2}$$

$$\text{where } \overline{\widehat{UQPE}_{X_1}^*}(\tau) = \frac{1}{B} \sum_{b=1}^B \widehat{UQPE}_{X_1}^{*b}(\tau).$$

4. Compute the $1 - \alpha$ confidence interval $[\widehat{UQPE}_{X_1}^{*[\alpha/2]}(\tau), \widehat{UQPE}_{X_1}^{*[1-\alpha/2]}(\tau)]$ using the ordered statistics of the bootstrap sample.

5 Monte Carlo experiments

This section presents several simulation exercises to study the finite sample performance of the proposed estimator. First, we assess the matching function estimator. Second, we evaluate the unconditional quantile partial effect (UQPE) estimation.

The first data generating process (DGP) we consider is as following:

$$y_i = 1 + x_i + (1 + \theta x_i)u_i, \tag{16}$$

where $x_i \sim N(10, 1)$ and u_i is a random variable with $E(u_i) = 0$, $V(u_i) = 1$ and independent of x_i . The distribution of u_i is specified below as either standard Gaussian or (standardized) Chi-squared with 1 degree of freedom. The parameter θ controls the type of effect of the covariate x on the distribution of $y|x$: when $\theta = 0$ the effect is a *location shift*, and if $\theta \neq 0$ is a *location-scale*

shift. In the former case the conditional quantile regression (CQR) effects are constant across quantiles, while in the latter case they vary.

Second, we use a DGP with an additional covariate

$$y_i = 1 + w_i + x_i + (1 + \theta x_i)u_i, \tag{17}$$

where we consider two cases: (i) $w_i \sim N(10, 1)$ (independent of x_i); (ii) $w_i = 10 + (x_i + N(10, 1) - 20)/\sqrt{2}$, where we make w_i correlated with x_i .

5.1 Matching function estimator

The proposed UQPE estimator relies on the estimator of the matching function for the quantiles, $\hat{\xi}_\tau(x)$. This subsection presents simulations exercises for assessing the accuracy of the matching estimator as given in equation (14). Recall from Example 1, equation (9), that in the simple linear case we have an explicit formula for the population matching function, $\xi_\tau(x)$. Thus, we are able to use simulations to assess the finite sample performance of the estimator.

We consider experiments using DGP model in (16) for a pure location model, $\theta = 0$, as well as a location-scale model, $\theta = 1$. We use $x_i \sim N(10, 1)$ and $u_i \sim N(10, 1)$. Each experiment has 100 simulations of the DGP with sample sizes $n = \{250, 500, 2500, 5000\}$, and quantile grid sizes $m = \{9, 24, 99, 199\}$, respectively. We consider three quantiles $\tau \in \{0.25, 0.50, 0.75\}$. Figure 4 reports results for the location case, and Figure 5 displays results for the location-scale case. In each figure, we plot the parameter of interest (the true value of the matching function), the estimates (average estimates over the number of simulations), as well as the 95% empirical confidence interval.¹⁰

Simulation results show evidence that the matching function estimator provides an approximately asymptotically unbiased estimator for both the pure location and location-scale models with a better performance of sample sizes of $n \geq 500$. Point estimates are close to the populations counterparts even for small samples and grids. As sample size and grid increase together, point estimates become very close to the population and confidence intervals shrink. Section A.5 of the appendix contains a proposal for conducting inference on the matching function should this be of interest.

5.2 UQPE estimation

Now we investigate the finite sample performance of the proposed UQPE estimator as in equation (15). In what follows, We label this estimator as Nadaraya-Watson (NW). For comparison, we also implement the RIF regression model for UQR for each unconditional quantile using the `rifvar` STATA command (Rios-Avila, 2020). In this case, the UQPE is estimated by OLS with a cubic polynomial model of the RIF for each quantile as a function of x_i (or as a function of x_i and

¹⁰These are computed as the 2.5-th and 97.5-th empirical percentiles of estimates across simulations.

Figure 4: Estimation of matching functions, $u \sim N(0, 1)$ and $\theta = 0$ (pure location).

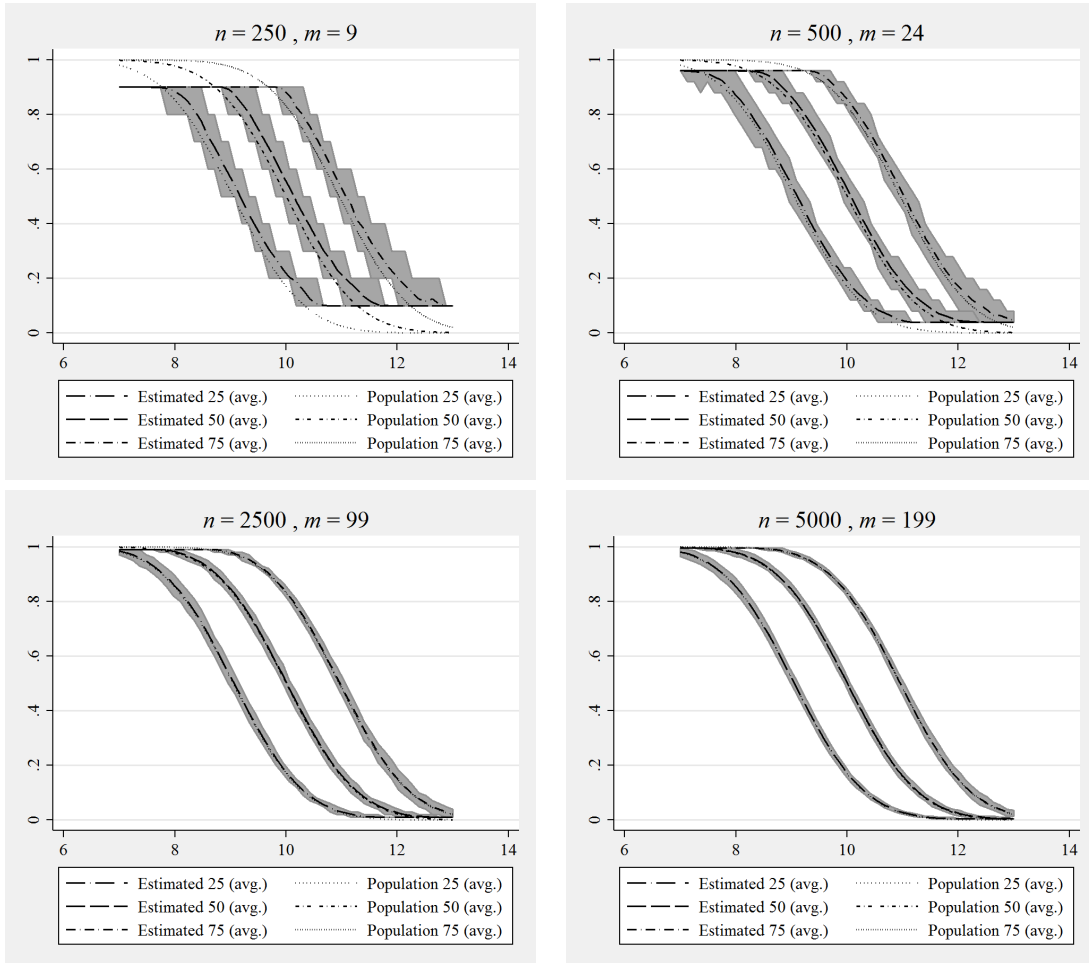
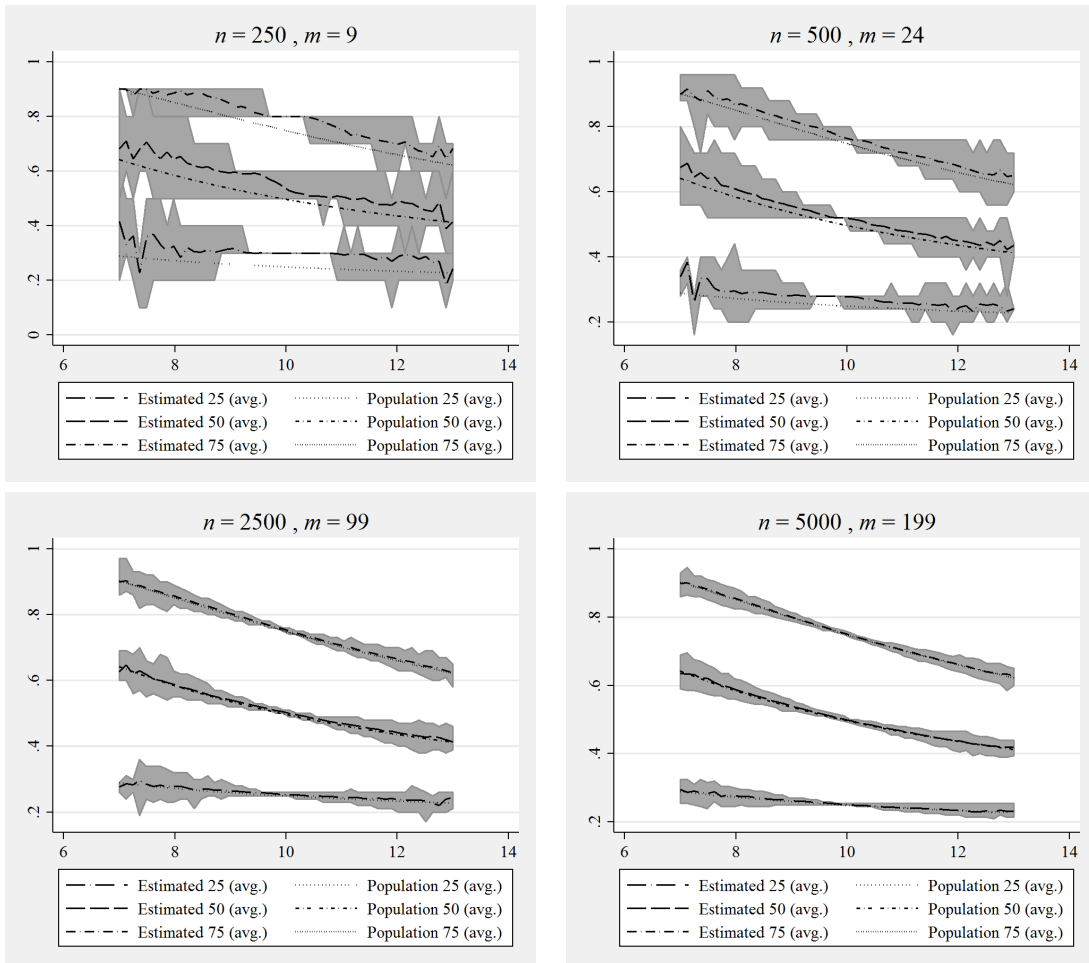


Figure 5: Estimation of matching functions, $u \sim N(0,1)$ and $\theta = 1$ (location-scale).



w_i for the model with additional covariates). Although not reported the cubic model overperforms the linear and quadratic implementation, available from the Authors upon request. Finally we also implement the RIF-Logit model suggested by [Firpo, Fortin, and Lemieux \(2009\)](#).

Each experiment has 1,000 simulations of the DGP with sample sizes $n = \{250, 500, 2500, 5000\}$, and quantiles grid sizes $m = \{9, 24, 99, 199\}$, respectively. We consider three quantiles $\tau \in \{0.25, 0.50, 0.75\}$. Moreover, we use a bandwidth $h_n = 0.9\hat{\sigma}_y n^{-1/6}$ and the 4th order Gaussian kernel function, $K(u) = \frac{3-u^2}{2}\phi(u)$, where $\phi(u)$ is the pdf of a standard normal, as discussed above. To evaluate the procedures we report the bias, variance, and mean-squared error (MSE).

Table 1 presents results for the baseline model for the simple location-shift model (i.e. $\theta = 0$) and Gaussian covariate and innovation. Both RIF models and NW estimators have a good performance in terms of bias, variance and MSE. These three statistics decrease for both estimators as sample size increases, for all three quantiles.

Tables 2 and 3 present simulations results for Gaussian and Chi-squared innovations, respectively, for the location-scale shift model (i.e. $\theta = 1$) with a Gaussian covariate. For all cases we observe that for the proposed NW estimator the bias and variance reduces as n increases. The relative performance to the RIF models varies depending on the simulation exercises, but in most cases either the NW estimator outperforms the RIF one.

Tables 4 and 5 collect simulation results for cases where there is an additional covariate, w_i . The former case uses an independent additional covariate and in the latter case w_i is correlated with x_i . In both cases we use the model with $\theta = 1$ and $x_i \sim N(10, 1)$. The results are also in line with previous ones, highlighting a good performance of the NW estimator in terms of bias, variance, and MSE.

Overall, these simulation results indicate that our proposed method produces a consistent estimator, where both bias and variance reduce as n increases. In some cases, however, the bias improvement applies only for $n \geq 500$.

Finally, Table 6 presents simulation exercises where we consider different bandwidth and kernels choices. In particular, we use $h_n = 0.9\hat{\sigma}_y n^{-1/4}$, $h_n = 0.9\hat{\sigma}_y n^{-1/5}$ and $h_n = 0.9\hat{\sigma}_y n^{-1/6}$. In order to explore our proposed optimal choice $h_n \propto n^{-1/6}$ we compare with $h_n \propto n^{-1/5}$, the standard bandwidth choice in nonparametric kernel estimators and also with $h_n \propto n^{-1/4}$. Next we also evaluate the proposed 4th order Gaussian kernel with the standard Gaussian one. We consider the location-scale model with $\theta = 1$ and Gaussian errors. The results show evidence that there are only small differences across bandwidths and kernels which suggest that the estimator is robust to these choices. For the empirical researcher this suggests that our proposed estimator can be combined with the standard nonparametric implementation.

Table 1: Location model: $\theta = 0$, and $x_i \sim N(10, 1)$, $u_i \sim N(0, 1)$.

Estimator		RIF-OLS (cubic)			RIF-Logit			NW		
τ	n	Bias	Variance	MSE	Bias	Variance	MSE	Bias	Variance	MSE
25	250	0.02470	0.01559	0.01620	0.02547	0.01736	0.01800	-0.00414	0.00455	0.00457
	500	0.02432	0.00876	0.00935	0.01837	0.00960	0.00994	0.00006	0.00241	0.00241
	2500	0.01299	0.00202	0.00219	0.00924	0.00224	0.00233	-0.00112	0.00046	0.00046
	5000	0.00866	0.00122	0.00129	0.00595	0.00136	0.00140	-0.00053	0.00022	0.00022
50	250	0.04550	0.01215	0.01422	0.03407	0.01344	0.01460	-0.00029	0.00421	0.00421
	500	0.03899	0.00736	0.00888	0.02971	0.00815	0.00903	0.00207	0.00219	0.00220
	2500	0.02319	0.00166	0.00220	0.01701	0.00188	0.00217	-0.00054	0.00043	0.00043
	5000	0.01764	0.00090	0.00121	0.01287	0.00103	0.00119	-0.00007	0.00020	0.00020
75	250	0.03803	0.01592	0.01737	0.02125	0.01714	0.01759	0.00235	0.00495	0.00496
	500	0.02561	0.00846	0.00911	0.01772	0.00920	0.00952	0.00390	0.00257	0.00259
	2500	0.01217	0.00219	0.00234	0.00856	0.00242	0.00249	0.00012	0.00047	0.00047
	5000	0.01135	0.00120	0.00133	0.00852	0.00136	0.00143	0.00044	0.00023	0.00023

Notes: Monte Carlo experiments based on 1000 simulations.

Table 2: Location-shift model: $\theta = 1$, and $x_i \sim N(10, 1)$, $u_i \sim N(0, 1)$.

Estimator		RIF-OLS (cubic)			RIF-Logit			NW		
τ	n	Bias	Variance	MSE	Bias	Variance	MSE	Bias	Variance	MSE
25	250	-0.00459	1.01506	1.01508	0.00897	0.97017	0.97025	0.14896	0.80540	0.82759
	500	0.02808	0.47673	0.47752	0.02558	0.46631	0.46696	0.09189	0.42565	0.43410
	2500	-0.00302	0.09242	0.09242	-0.00503	0.09132	0.09135	0.00395	0.08536	0.08538
	5000	0.00355	0.04448	0.04449	0.00239	0.04412	0.04413	0.00616	0.04157	0.04161
50	250	0.05602	0.86653	0.86967	0.04100	0.81673	0.81841	0.11362	0.68828	0.70119
	500	0.07079	0.43029	0.43530	0.05600	0.41888	0.42201	0.07073	0.35817	0.36317
	2500	0.01198	0.08004	0.08018	0.00445	0.07946	0.07948	0.00298	0.06983	0.06984
	5000	0.01055	0.03963	0.03974	0.00541	0.03937	0.03940	-0.00072	0.03342	0.03342
75	250	0.07899	1.06619	1.07243	0.06986	0.97638	0.98126	0.19854	0.82577	0.86519
	500	0.04227	0.50446	0.50625	0.03024	0.49447	0.49538	0.09237	0.42463	0.43316
	2500	0.03413	0.10107	0.10223	0.02793	0.10136	0.10214	0.02208	0.08080	0.08129
	5000	0.02322	0.04809	0.04863	0.01820	0.04827	0.04860	0.01518	0.03739	0.03762

Notes: Monte Carlo experiments based on 1000 simulations.

Table 3: Location-scale shift model: $\theta = 1$, and $x_i \sim N(10, 1)$, $u_i \sim (\chi_1^2 - 1)/\sqrt{2}$.

Estimator		RIF-OLS (cubic)			RIF-Logit			NW		
τ	n	Bias	Variance	MSE	Bias	Variance	MSE	Bias	Variance	MSE
25	250	0.38010	0.12921	0.27369	0.29658	0.10312	0.19108	0.04788	0.05469	0.05698
	500	0.31075	0.05070	0.14727	0.24543	0.04069	0.10093	0.01956	0.02024	0.02062
	2500	0.18007	0.00637	0.03880	0.13126	0.00527	0.02250	0.01161	0.00316	0.00330
	5000	0.13945	0.00289	0.02234	0.09669	0.00240	0.01175	0.00808	0.00157	0.00164
50	250	-0.07688	0.22020	0.22611	-0.08321	0.20770	0.21463	0.08704	0.34568	0.35325
	500	-0.08876	0.10207	0.10995	-0.08481	0.10242	0.10961	0.03112	0.14878	0.14975
	2500	-0.08059	0.02002	0.02652	-0.06304	0.02133	0.02531	0.00525	0.02376	0.02379
	5000	-0.06959	0.01075	0.01559	-0.05158	0.01150	0.01416	-0.00256	0.01236	0.01237
75	250	-0.05772	1.39024	1.39357	-0.03968	1.40400	1.40557	0.23834	1.94604	2.00285
	500	-0.08805	0.70950	0.71725	-0.06975	0.72037	0.72524	0.04383	0.80686	0.80878
	2500	-0.02743	0.14136	0.14211	-0.02110	0.14339	0.14384	0.01597	0.13421	0.13446
	5000	-0.03405	0.06550	0.06666	-0.02846	0.06630	0.06711	-0.00658	0.05981	0.05986

Notes: Monte Carlo experiments based on 1000 simulations.

Table 4: Location-scale shift model with independent covariate: $\theta = 1$, $w_i \sim N(10, 1)$, and $x_i \sim N(10, 1)$, $u_i \sim N(0, 1)$.

Estimator		RIF-OLS (cubic)			RIF-Logit			NW		
τ	n	Bias	Variance	MSE	Bias	Variance	MSE	Bias	Variance	MSE
25	250	-0.01793	0.96720	0.96752	-0.01313	0.90922	0.90940	0.11615	0.76633	0.77982
	500	0.02157	0.52336	0.52383	0.01794	0.51131	0.51163	0.08909	0.40590	0.41384
	2500	0.00591	0.09537	0.09541	0.00476	0.09394	0.09396	0.02003	0.08346	0.08386
	5000	-0.00026	0.04463	0.04463	-0.00131	0.04433	0.04433	0.00425	0.03926	0.03928
50	250	0.00228	0.93384	0.93384	-0.02212	0.87770	0.87819	0.07196	0.68486	0.69004
	500	0.04824	0.41277	0.41510	0.03867	0.39961	0.40110	0.06137	0.32540	0.32917
	2500	0.00705	0.08055	0.08060	0.00002	0.07976	0.07976	-0.00710	0.07042	0.07047
	5000	-0.00166	0.03845	0.03846	-0.00712	0.03802	0.03807	-0.01221	0.03167	0.03182
75	250	-0.00581	1.01975	1.01979	-0.01927	0.98074	0.98112	0.10075	0.86558	0.87573
	500	0.04783	0.45838	0.46066	0.03623	0.45086	0.45217	0.05951	0.37254	0.37609
	2500	0.01022	0.10191	0.10201	0.00445	0.10232	0.10234	0.00657	0.07819	0.07823
	5000	-0.00166	0.04808	0.04808	-0.00545	0.04845	0.04848	-0.01084	0.03872	0.03884

Notes: Monte Carlo experiments based on 1000 simulations.

Table 5: Location-shift model with correlated covariate: $\theta = 1$, $w_i = 10 + (x_i + N(10, 1) - 20)/\sqrt{2}$, and $x_i \sim N(10, 1)$, $u_i \sim N(0, 1)$.

Estimator		RIF-OLS (cubic)			RIF-Logit			NW		
τ	n	Bias	Variance	MSE	Bias	Variance	MSE	Bias	Variance	MSE
25	250	-0.09324	2.05569	2.06438	-0.06573	1.96614	1.97046	0.04767	1.61287	1.61514
	500	-0.02007	0.98728	0.98769	-0.02275	0.96771	0.96823	0.04294	0.78275	0.78459
	2500	-0.02730	0.17002	0.17077	-0.02856	0.16783	0.16864	-0.01554	0.14793	0.14817
	5000	-0.01900	0.08653	0.08689	-0.02042	0.08588	0.08630	-0.01607	0.07613	0.07639
50	250	0.00075	1.90333	1.90334	-0.02501	1.82672	1.82735	0.06929	1.42991	1.43471
	500	0.03006	0.82247	0.82337	0.01738	0.79296	0.79326	0.02890	0.65508	0.65591
	2500	-0.01982	0.15795	0.15834	-0.02668	0.15681	0.15752	-0.02458	0.13300	0.13360
	5000	-0.00523	0.07573	0.07576	-0.01103	0.07484	0.07497	-0.01595	0.06313	0.06339
75	250	0.00633	2.04659	2.04663	-0.01486	1.92153	1.92175	0.12280	1.62456	1.63964
	500	0.04492	1.03865	1.04067	0.03612	1.02019	1.02150	0.07235	0.78620	0.79143
	2500	0.01423	0.19626	0.19646	0.00868	0.19582	0.19590	0.02095	0.15078	0.15122
	5000	0.00299	0.09886	0.09887	-0.00072	0.09855	0.09855	-0.00338	0.07759	0.07760

Notes: Monte Carlo experiments based on 1000 simulations.

Table 6: Kernel and bandwidth choice ($\theta = 1$ and $u_i \sim N(0, 1)$).

Kernel	τ	n	Bias			Variance			ECM		
			$n^{-1/4}$	$n^{-1/5}$	$n^{-1/6}$	$n^{-1/4}$	$n^{-1/5}$	$n^{-1/6}$	$n^{-1/4}$	$n^{-1/5}$	$n^{-1/6}$
Gauss	25	250	0.14894	0.14868	0.14849	0.80470	0.80311	0.80181	0.82688	0.82522	0.82386
		500	0.09184	0.09142	0.09108	0.42566	0.42411	0.42300	0.43409	0.43246	0.43130
		2500	0.00411	0.00379	0.00359	0.08542	0.08527	0.08516	0.08544	0.08528	0.08517
		5000	0.00629	0.00602	0.00585	0.04160	0.04154	0.04150	0.04164	0.04157	0.04153
	50	250	0.11349	0.11159	0.11028	0.68763	0.68532	0.68358	0.70051	0.69778	0.69574
		500	0.07096	0.06935	0.06825	0.35794	0.35727	0.35668	0.36298	0.36208	0.36133
		2500	0.00308	0.00249	0.00201	0.06978	0.06978	0.06975	0.06979	0.06978	0.06975
		5000	-0.00071	-0.00108	-0.00144	0.03340	0.03341	0.03340	0.03340	0.03341	0.03341
75	250	0.19834	0.19632	0.19488	0.82528	0.82485	0.82442	0.86462	0.86339	0.86239	
	500	0.09263	0.09177	0.09123	0.42418	0.42507	0.42552	0.43276	0.43349	0.43384	
	2500	0.02237	0.02224	0.02228	0.08078	0.08093	0.08107	0.08128	0.08143	0.08156	
	5000	0.01525	0.01541	0.01558	0.03740	0.03744	0.03750	0.03763	0.03768	0.03774	
25	250	0.14930	0.14916	0.14896	0.80707	0.80633	0.80540	0.82936	0.82857	0.82759	
	500	0.09245	0.09215	0.09189	0.42805	0.42668	0.42565	0.43660	0.43517	0.43410	
	2500	0.00452	0.00417	0.00395	0.08561	0.08547	0.08536	0.08563	0.08549	0.08538	
	5000	0.00661	0.00632	0.00616	0.04167	0.04161	0.04157	0.04171	0.04165	0.04161	
Gauss Mod	50	250	0.11667	0.11492	0.11362	0.69141	0.68978	0.68828	0.70502	0.70299	0.70119
		500	0.07343	0.07188	0.07073	0.35882	0.35857	0.35817	0.36421	0.36374	0.36317
	75	2500	0.00373	0.00331	0.00298	0.06978	0.06981	0.06983	0.06979	0.06982	0.06984
		5000	-0.00038	-0.00054	-0.00072	0.03339	0.03341	0.03342	0.03339	0.03341	0.03342
5000	250	0.20159	0.20011	0.19854	0.82606	0.82631	0.82577	0.86670	0.86635	0.86519	
	500	0.09393	0.09311	0.09237	0.42264	0.42400	0.42463	0.43146	0.43267	0.43316	
	2500	0.02257	0.02229	0.02208	0.08065	0.08070	0.08080	0.08116	0.08120	0.08129	
	5000	0.01509	0.01517	0.01518	0.03740	0.03737	0.03739	0.03762	0.03760	0.03762	

Notes: Monte Carlo experiments based on 1000 simulations.

6 Empirical Application

This section illustrates the UQPE estimator with an analysis of Engel’s curves. The original concept corresponds to Ernst Engel (1857, cited in [Koenker \(2005\)](#), pp. 78-79) who studied the European working class households consumption in the 19th century. Engel curves describe how household expenditures on particular goods and services depend on household income. The analysis of Engel curves has a long history of estimating the expenditure-income relationship. An empirical result commonly referred to as “Engel’s law” states that the poorer a family is, the larger the budget share it spends on food. Other categories of expenditure present a less robust pattern. Hence, we investigate the hypothesis that food expenditure constitutes a declining share of household income.¹¹

We apply this framework to household expenditures in Argentina using the national survey of expenditures (Encuesta Nacional de Gasto de los Hogares, known as ENGHO 2017-2018), implemented by the Instituto Nacional de Estadística y Censos (INDEC). The survey was carried out between November 2017 and November 2018. The ENGHO 2017-2018 surveys the households’ living conditions in terms of their access to goods and services, as well as their income. The data contains information about household expenditures on different goods and services. About 21,547 households were randomly selected and participated on the survey. We consider both food household expenditures and total non-durable consumption for comparison.¹²

We estimate both UQPE and CQPE. The former analysis corresponds to evaluating effect of an increase in income for every household in a uniform pattern on the unconditional quantile of food expenditure while focusing on the entire distribution of expenditure. The latter effect corresponds to the study of how expenditure changes when marginally increasing income conditional on income. For comparison, we also provide estimate results for the RIF regression and RIF-Logit of [Firpo, Fortin, and Lemieux \(2009\)](#). We use both income and expenditures in logarithm, so that the coefficient estimates can be interpreted as an elasticity. Confidence intervals are computed using 200 bootstrap replications.

We estimate these models for different quantiles. The results are collected in [Table 7](#) and [Figure 6](#) for food expenditures, and [Table 8](#) and [Figure 8](#) for total non-durable expenditures. In order to explore the results in more detail, we also plot the by-product of this analysis that is the matching quantile function. This was implicitly used for the estimation of the UQPE NW estimator. [Figures 7](#) and [9](#) plot the estimated match for $\tau = \{0.25, 0.50, 0.75\}$ for different values

¹¹There is a large literature on empirical applications of the Engel’s curve, see, e.g., for example, among many others, [Lewbel \(1997, 2008\)](#), [Blundell, Chen, and Kristensen \(2007\)](#), [Chai and Moneta \(2010\)](#), [Chernozhukov, Fernández-Val, and Kowalski \(2015\)](#).

¹²In particular, the sample has information on: (i) food and non-alcoholic beverages, (ii) alcoholic beverages and tobacco, (iii) clothing and footwear, (iv) housing, water, electricity, gas and other fuels, (v) home equipment and maintenance, (vi) health, (vii) transportation, (viii) communications, (ix) recreation and culture, (x) education, (xi) restaurants and hotels, and (xii) miscellaneous goods and services. Both expenses and income are transformed to represent monthly values. Since the monetary values of each household are expressed in current currency at the time of the survey, an inflation adjustment was made to transform them into constant currency for December of the fourth quarter of 2018 using the Consumer Price Index (CPI) computed by national statistical office, INDEC.

Table 7: Engel's curve for food expenditures.

	Quantile Partial Effect				
	10	25	50	75	90
<i>Conditional distribution</i>					
CQR	0.383*** (0.000571)	0.407*** (0.000422)	0.408*** (0.000246)	0.408*** (0.000278)	0.425*** (0.000336)
<i>Unconditional distribution</i>					
RIF (linear model)	0.367*** (0.0285)	0.388*** (0.0170)	0.427*** (0.0139)	0.396*** (0.0130)	0.393*** (0.0181)
RIF (quadratic model)	0.360*** (0.0275)	0.383*** (0.0166)	0.427*** (0.0140)	0.403*** (0.0129)	0.406*** (0.0182)
RIF (cubic model)	0.370*** (0.0279)	0.394*** (0.0169)	0.440*** (0.0143)	0.415*** (0.0137)	0.412*** (0.0183)
RIF (Logit)	0.327*** (0.0373)	0.395*** (0.0260)	0.434*** (0.0234)	0.403*** (0.0242)	0.427*** (0.0316)
NW	0.395*** (0.0166)	0.405*** (0.0111)	0.408*** (0.00851)	0.409*** (0.00809)	0.410*** (0.00870)
Observations	21,017	21,017	21,017	21,012	21,017

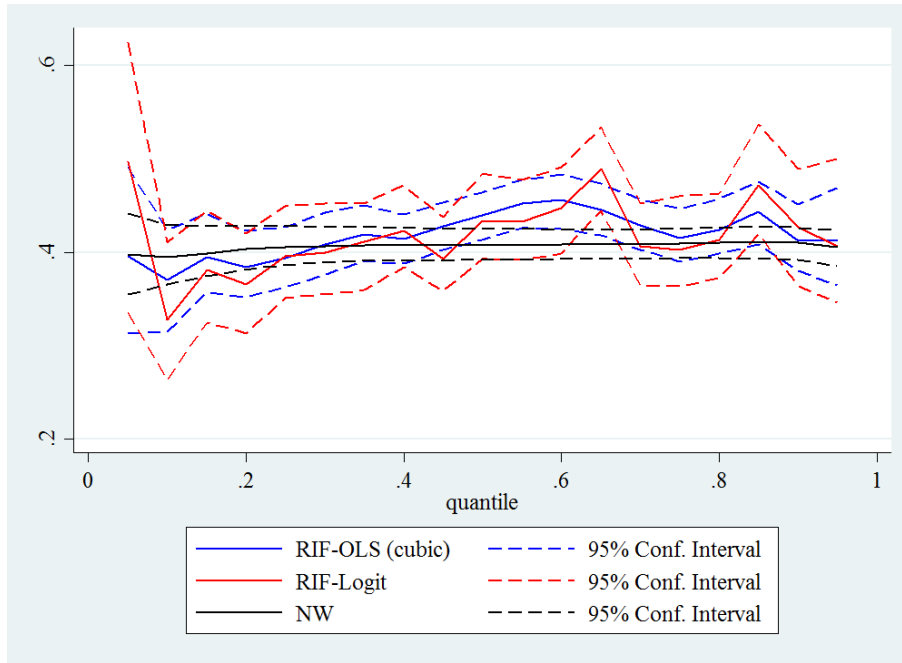
Notes: The CQR analysis corresponds to a regression of log food expenditures on log income. UQPE estimates the effect of a marginal change in log income on the unconditional distribution of log food expenditures. Standard errors in parentheses (analytical for CQR, bootstrap with 200 replications for RIF and NW). * indicates significance at 10 %, ** at 5 % and *** at 1 %.

Table 8: Engel's curve for total non-durable expenditures

	Quantile Partial Effect				
	10	25	50	75	90
<i>Conditional distribution</i>					
CQR	0.778*** (0.000359)	0.784*** (0.000265)	0.776*** (0.000211)	0.738*** (0.000283)	0.662*** (0.000295)
<i>Unconditional distribution</i>					
RIF (linear model)	0.700*** (0.0295)	0.706*** (0.0194)	0.761*** (0.0216)	0.768*** (0.0229)	0.731*** (0.0292)
RIF (quadratic model)	0.674*** (0.0278)	0.693*** (0.0183)	0.765*** (0.0217)	0.792*** (0.0251)	0.769*** (0.0338)
RIF (cubic model)	0.683*** (0.0267)	0.719*** (0.0183)	0.801*** (0.0217)	0.817*** (0.0229)	0.774*** (0.0275)
RIF (Logit)	0.622*** (0.0471)	0.641*** (0.0329)	0.755*** (0.0434)	0.794*** (0.0475)	0.811*** (0.0542)
NW	0.774*** (0.0142)	0.772*** (0.0102)	0.760*** (0.00837)	0.731*** (0.00855)	0.697*** (0.0103)
Observations	21,461	21,461	21,461	21,461	21,461

Notes: The CQR analysis corresponds to a regression of log non-durable expenditures on log income. UQPE estimates the effect of a marginal change in log income on the unconditional distribution of log non-durable expenditures. Standard errors in parentheses (analytical for CQR, bootstrap with 200 replications for RIF and NW). * indicates significance at 10 %, ** at 5 % and *** at 1 %.

Figure 6: Engel's curves for food expenditures



Notes: UQPE NW (black), RIF-OLS (cubic polynomial, blue) and RIF-Logit (red) estimates together with 95% confidence intervals estimated using bootstrap with 200 replications.

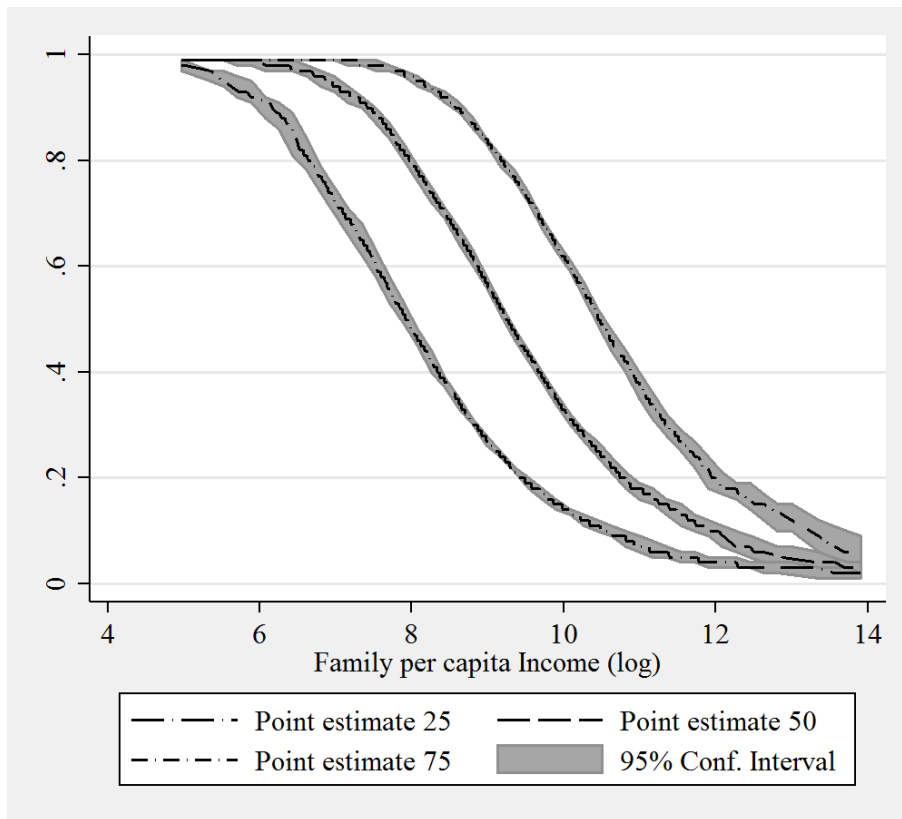
of log income, for food and non-durable expenditures, respectively. The last figures illustrate that for each τ there is a full range of variation in the corresponding CQPE model indexed by η .

The results for food expenditures show evidence that CQR coefficients are roughly constant across η , although mildly increasing. The proposed UQPE NW estimator is then also roughly constant across τ . The RIF estimates also have this pattern although they are estimated in a less precise manner. In all cases, the estimated effects can be interpreted as elasticities, implying that a 1% increase in income increase food consumption in less than 1%, about 0.4%. Moreover, since the CQR coefficients are mildly increasing, the variation in the UQPE has to be coming from the variation in the density of X given $Y = Q_\tau[Y]$. As τ increases, for the UQPE to increase, higher CQR coefficients must be getting higher weight. This happens if the density of “income | food = $Q_\tau[\text{food}]$ ” is moving to the right.

For the case of non-durable expenditures, the RIF estimates are increasing along τ , while the UQPE NW estimator is decreasing. The fact that RIF estimates have a larger range of variation than CQR and that it gives the counter-intuitive increasing pattern suggest that it might be misspecified. That is, if we assume that richest families have a higher saving rate than the poorer ones, the unconditional estimates should not increase along the quantiles. The analytical results used here indicate that the UQPE is a weighted average of CQPE, and as such, it cannot exceed its conditional heterogeneity. Our UQPE NW estimates clearly remain within the CQR variation, while RIF estimates do not.

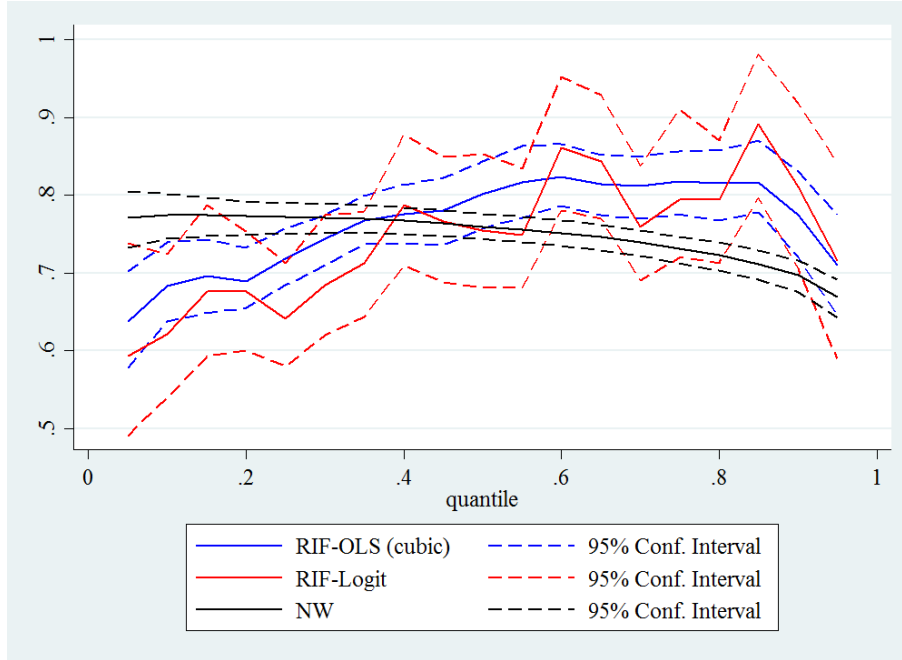
A take away note from this example is that the empirical researcher should be aware of

Figure 7: Estimated matching function - Food expenditures



Notes: Matched coefficients $\hat{\beta}_1(\hat{\xi}_\tau(\text{income}_i))$ for $\tau \in \{0.25, 0.50, 0.75\}$ estimates together with 95% confidence intervals estimated using bootstrap with 200 replications.

Figure 8: Engel’s curves for total non-durable expenditures.



Notes: UQPE NW (black), RIF-OLS (cubic polynomial, blue) and RIF-Logit (red) estimates together with 95% confidence intervals estimated using bootstrap with 200 replications).

different modelling choices. RIF and UQPE NW estimators both rely on different assumptions to estimate the same parameter. As such, it may be a good procedure to report different alternatives and to highlight if differences occur.

7 Conclusion

This paper considers the use of conditional quantile regression analysis to estimate unconditional quantile partial effects. The proposed methodology is based on a matching and reweighting result to link the unconditional effects to the conditional ones. This method thus benefits from the usual conditional quantile regression estimation techniques, and suggests a two-step estimator for the unconditional effects. In the first step one estimates a structural quantile regression model, and in the second step a nonparametric regression is applied. We establish the asymptotic properties of the estimator. Monte Carlo simulations show evidence that the estimator has good finite sample performance and is robust to the selection of bandwidth and kernel. To illustrate the proposed methods, we study Engel’s curves in Argentina.

The current paper can be extended in several directions. First, the proposed model uses a simple linear quantile regression framework and is based on its coefficient estimators. The current framework can be applied to any other \sqrt{n} -consistent estimation procedure. In particular, as an example, instrumental variables quantile regression and/or panel data models deliver consistent estimators for the conditional effects in several related statistical models. The current method-

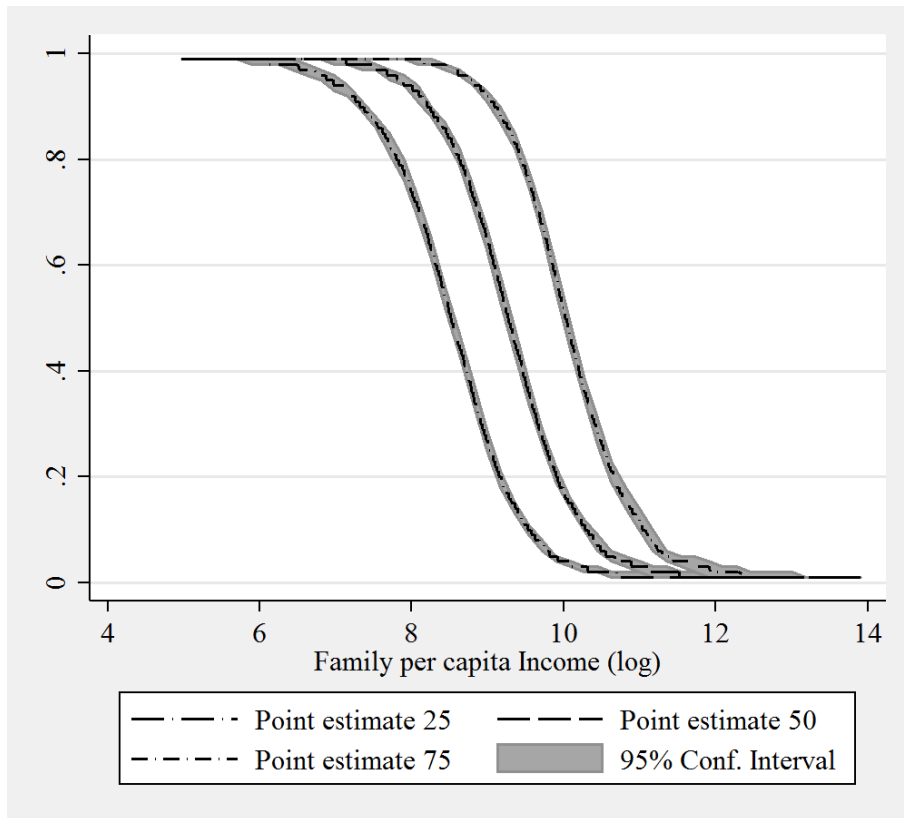


Figure 9: Estimated matching function - Non-durable expenditures

Notes: Matched coefficients $\hat{\beta}_1(\hat{\xi}_\tau(\text{income}_i))$ for $\tau \in \{0.25, 0.50, 0.75\}$ estimates together with 95% confidence intervals estimated using bootstrap with 200 replications.

ology could be extended to evaluate unconditional effects, starting from any initial consistent conditional estimation procedure. Second, the current proposed framework can be used to evaluate any other functional analysis related to the unconditional quantile regression one. In other words, to recover general distributional effects. Third, the Nadaraya-Watson estimator is the first approximation to a larger family of estimators that can be used to estimate the unconditional effects. Local linear regression models is a proposed refinement to obtain possibly better asymptotic properties. Finally, the analysis may be extended in formalizing the resampling procedure and studying to practically construct uniform confidence bands.

Appendix A

A.1 Proof of Lemma 1

Recall that $X = (X_1, X_2)'$, $x = (x_1, x_2)'$, and X_1 and x_1 are one-dimensional. First we show that $CQPE_{X_1}(\eta, x)$, defined in (2) as

$$CQPE_{X_1}(\eta, x) := \left. \frac{\partial Q_Y[\tau | X_1 = z, X_2 = x_2]}{\partial z} \right|_{z=x_1}$$

can be written in the way of (5) as

$$CQPE_{X_1}(\eta, x) = - \left. \frac{1}{f_{Y|X}(Q_Y[\tau | X = x] | x)} \frac{\partial F_{Y|X}(Q_Y[\tau | X = x] | z, x_2)}{\partial z} \right|_{z=x_1}.$$

By definition of quantiles, we have that this identity holds for all $x \in \mathcal{X}$ for a given fixed τ :

$$F_{Y|X}(Q_Y[\tau | X = x] | x) = \tau.$$

Differentiating both sides with respect to x_1 , we obtain

$$f_{Y|X}(Q_Y[\tau | X = x] | x) CQPE_{X_1}(\eta, x) + \left. \frac{\partial F_{Y|X}(Q_Y[\tau | X = x] | x)}{\partial z} \right|_{z=x_1} = 0.$$

Since $f_{Y|X}(Q_Y[\tau | X = x] | x) \neq 0$, then the result follows by solving for $CQPE_{X_1}(\eta, x)$. Since the matching is a singleton, then for every x , and any τ , we have $Q_Y[\xi_\tau(x) | X = x] = Q_Y[\tau]$. Thus, we evaluate $CQPE_{X_1}(\eta, x)$ at $\tau = \xi_\tau(x)$ to yield

$$CQPE_{X_1}(\xi_\tau(x), x) = - \left. \frac{1}{f_{Y|X}(Q_Y[\tau] | x)} \frac{\partial F_{Y|X}(Q_Y[\tau] | z, x_2)}{\partial z} \right|_{z=x_1}.$$

Given the identification result for $UQPE_{X_1}(\tau)$ in equation (4), we have that

$$UQPE_{X_1}(\tau) = \int CQPE_{X_1}(\xi_\tau(x), x) \frac{f_{Y|X}(Q_Y[\tau] | x)}{f_Y(Q_Y[\tau])} dF_X(x),$$

which is the result in (7). Since f_Y and f_X are non-zero, then

$$\frac{f_{Y|X}(Q_Y[\tau] | x)}{f_Y(Q_Y[\tau])} f_X(x) = \frac{f_{Y,X}(Q_Y[\tau], x)}{f_Y(Q_Y[\tau]) f_X(x)} f_X(x) = f_{X|Y}(x | Q_Y[\tau]).$$

Therefore, we obtain (8):

$$UQPE_{X_1}(\tau) = E[CQPE_{X_1}(\xi_\tau(X), X) | Y = Q_Y[\tau]].$$

A.2 Proof of Theorem 1

Let

$$\Psi_\tau(\eta|x) = Q_Y[\eta|X = x] - Q_Y[\tau].$$

Here, τ and x are fixed, and the criterion function is the map $[\epsilon, 1 - \epsilon] \ni \eta \mapsto \Psi_\tau(\eta|x)$ for $0 < \epsilon < 1/2$. Under Assumption 1.2, $y \mapsto F_{Y|X}(y|x)$ is strictly increasing, and hence $\Psi_\tau(\eta|x)$ has a unique zero given by $\xi_\tau(x) = F_{Y|X}(Q_\tau[Y]|x)$. This shows that $Q_\tau[Y]$ is $\xi_\tau(x)$ -conditional quantile of $Y|X = x$. By Assumption 1.1, this can be written as $x'\beta(\xi_\tau(x)) = Q_\tau[Y]$.

Now we will show consistency: $\hat{\xi}_\tau(x) \xrightarrow{p} \xi_\tau(x)$. The matching function is defined to be the (approximate) zero of the random criterion function $\Psi_{\tau,n}(\eta|x)$:

$$\Psi_{\tau,n}(\hat{\xi}_\tau(x)|x) = x'\hat{\beta}(\hat{\xi}_\tau(x)) - \hat{Q}_Y[\tau].$$

Indeed, the computational procedure outlined in equation (14) implicitly defines $\hat{\xi}_\tau(x)$ as η_j for some j in $\{1, 2, \dots, m\}$ such that

$$x'\hat{\beta}(\eta_j) \leq \hat{Q}_Y[\tau] < x'\hat{\beta}(\eta_{j+1}).$$

We want to show that this, together with Assumption 1.5 that ensures $\Delta\eta = \eta_{j+1} - \eta_j = o(n^{-1/2})$, imply that $\Psi_{\tau,n}(\hat{\xi}_\tau(x)|x) = o_p(n^{-1/2})$. For a given n , let $\eta_j = \hat{\xi}_\tau(x)$, that is

$$\begin{aligned} x'\hat{\beta}(\eta_j) &\leq \hat{Q}_Y[\tau] < x'\hat{\beta}(\eta_{j+1}) \\ 0 &\leq \hat{Q}_Y[\tau] - x'\hat{\beta}(\eta_j) < x'\hat{\beta}(\eta_{j+1}) - x'\hat{\beta}(\eta_j) \\ 0 &\leq -\Psi_{\tau,n}(\hat{\xi}_\tau(x)|x) < x'(\hat{\beta}(\eta_{j+1}) - \hat{\beta}(\eta_j)). \end{aligned}$$

We focus on the difference $\hat{\beta}(\eta_{j+1}) - \hat{\beta}(\eta_j)$. We write

$$\begin{aligned} \hat{\beta}(\eta_{j+1}) - \hat{\beta}(\eta_j) &= \hat{\beta}(\eta_{j+1}) - \beta(\eta_{j+1}) - (\hat{\beta}(\eta_j) - \beta(\eta_j)) \\ &\quad + \beta(\eta_{j+1}) - \beta(\eta_j). \end{aligned}$$

For the last term, we can write $\beta(\eta_{j+1}) - \beta(\eta_j) = \beta'(\tilde{\eta})(\eta_{j+1} - \eta_j) = o(n^{-1/2})$ because the derivative is bounded by Assumption 1.3. To alleviate notation, define:

$$J(\eta) = E [f_{Y|X}(X'\beta(\eta)|X)XX'],$$

which is differentiable with bounded derivative by 1.3. Using Assumption 1.3 we have that

$$\hat{\beta}(\eta) - \beta(\eta) = J(\eta)^{-1} \frac{1}{n} \sum_{i=1}^n (\eta - \mathbb{1}\{y_i \leq x'_i\beta(\eta)\}) x_i + o_p(n^{-1/2}),$$

so that

$$\begin{aligned}
\hat{\beta}(\eta_{j+1}) - \beta(\eta_{j+1}) - (\hat{\beta}(\eta_j) - \beta(\eta_j)) &= J(\eta_{j+1})^{-1} \frac{1}{n} \sum_{i=1}^n (\eta_{j+1} - \mathbb{1}\{y_i \leq x_i' \beta(\eta_{j+1})\}) x_i \\
&\quad - J(\eta_j)^{-1} \frac{1}{n} \sum_{i=1}^n (\eta_j - \mathbb{1}\{y_i \leq x_i' \beta(\eta_j)\}) x_i + o_p(n^{-1/2}) \\
&= J(\eta_{j+1})^{-1} \frac{1}{n} \sum_{i=1}^n \left[(\eta_{j+1} - \mathbb{1}\{y_i \leq x_i' \beta(\eta_{j+1})\}) x_i \right. \\
&\quad \left. - (\eta_j - \mathbb{1}\{y_i \leq x_i' \beta(\eta_j)\}) x_i \right] \\
&\quad + \left(J(\eta_{j+1})^{-1} - J(\eta_j)^{-1} \right) \frac{1}{n} \sum_{i=1}^n (\eta_j - \mathbb{1}\{y_i \leq x_i' \beta(\eta_j)\}) x_i + o_p(n^{-1/2}).
\end{aligned}$$

By Assumption 1.2, $J(\eta)$ is bounded away from zero, so that $J(\eta)^{-1}$ is bounded. We focus first on the difference in the sums.

$$\frac{1}{n} \sum_{i=1}^n (\eta_{j+1} - \mathbb{1}\{y_i \leq x_i' \beta(\eta_{j+1})\}) - \eta_j + \mathbb{1}\{y_i \leq x_i' \beta(\eta_j)\}) x_i.$$

We note first that since $\eta_j < \eta_{j+1}$, by definition of quantiles, $x_i' \beta(\eta_j) < x_i' \beta(\eta_{j+1})$. This means that if $\mathbb{1}\{y_i \leq x_i' \beta(\eta_{j+1})\} = 0$, then $\mathbb{1}\{y_i \leq x_i' \beta(\eta_j)\} = 0$, and if $\mathbb{1}\{y_i \leq x_i' \beta(\eta_{j+1})\} = 1$, then either $\mathbb{1}\{y_i \leq x_i' \beta(\eta_j)\} = 0$, or $\mathbb{1}\{y_i \leq x_i' \beta(\eta_j)\} = 1$. Thus, the difference

$$\mathbb{1}\{y_i \leq x_i' \beta(\eta_{j+1})\} - \mathbb{1}\{y_i \leq x_i' \beta(\eta_j)\}$$

is either 1 or 0. Using this, we have

$$\begin{aligned}
&\left| \frac{1}{n} \sum_{i=1}^n (\eta_{j+1} - \mathbb{1}\{y_i \leq x_i' \beta(\eta_{j+1})\}) - \eta_j + \mathbb{1}\{y_i \leq x_i' \beta(\eta_j)\}) x_i \right| \\
&\leq \frac{1}{n} \sum_{i=1}^n |\eta_{j+1} - \mathbb{1}\{y_i \leq x_i' \beta(\eta_{j+1})\} - \eta_j + \mathbb{1}\{y_i \leq x_i' \beta(\eta_j)\}| |x_i| \\
&\leq |\eta_{j+1} - \eta_j| \frac{1}{n} \sum_{i=1}^n |x_i| \\
&= o(n^{-1/2}) O_p(1) = o_p(n^{-1/2}).
\end{aligned}$$

Now, for the second term, we note that

$$\begin{aligned} \left| \left(J(\eta_{j+1})^{-1} - J(\eta_j)^{-1} \right) \frac{1}{n} \sum_{i=1}^n (\eta_j - \mathbb{1} \{y_i \leq x'_i \beta(\eta_j)\}) x_i \right| &\leq \left| \frac{J(\eta_j) - J(\eta_{j+1})}{J(\eta_j)J(\eta_{j+1})} \right| \frac{1}{n} \sum_{i=1}^n |x_i| \\ &= \left| \frac{J'(\tilde{\eta})(\eta_{j+1} - \eta_j)}{J(\eta_j)J(\eta_{j+1})} \right| \frac{1}{n} \sum_{i=1}^n |x_i| \\ &= o(n^{-1/2})O_p(1) = o_p(n^{-1/2}). \end{aligned}$$

where $\tilde{\eta}$ is between η_j and η_{j+1} . Therefore, we have that

$$\Psi_{\tau,n}(\hat{\xi}_\tau(x)|x) = o_p(n^{-1/2}).$$

Since $\hat{\xi}_\tau(x)$ is a Z-estimator, we follow Theorem 5.9 in [van der Vaart \(1998\)](#). We need to show (i) that the criterion function converges uniformly in probability:

$$\sup_{\eta \in [\epsilon, 1-\epsilon]} |\Psi_{\tau,n}(\eta|x) - \Psi_\tau(\eta|x)| \xrightarrow{p} 0, \quad (\text{A.1})$$

and (ii) that the zero is well-separated: for any $\Delta > 0$

$$\inf_{\eta: |\eta - \xi_\tau(x)| > \Delta} |\Psi_\tau(\eta|X)| > 0.$$

To show (A.1), we note that by [Assumption 1.1](#)

$$\begin{aligned} \sup_{\eta \in [\epsilon, 1-\epsilon]} |\Psi_{\tau,n}(\eta|x) - \Psi_\tau(\eta|x)| &= \sup_{\eta \in [\epsilon, 1-\epsilon]} |x' \hat{\beta}(\eta) - \hat{Q}_Y[\tau] - x' \beta(\eta) + Q_Y[\tau]| \\ &\leq \|x\| \sup_{\eta \in [\epsilon, 1-\epsilon]} |\hat{\beta}(\eta) - \beta(\eta)| + |\hat{Q}_Y[\tau] - Q_Y[\tau]| \\ &= \|x\| O_p(n^{-1/2}) + O_p(n^{-1/2}) \\ &= o_p(1). \end{aligned}$$

where $\|\cdot\|$ is the Euclidean norm, and the bounds follow from [Assumptions 1.3](#) and [1.4](#). To show that the zero is well-separated, we note that by [Assumption 1.2](#), $y \mapsto F_{Y|X}(y|x)$ is strictly increasing, so that for any $\Delta > 0$, if $|\eta - \xi_\tau(x)| > \Delta$, then by the mean-value theorem

$$|\eta - F_{Y|X}(Q_\tau[Y]|x)| = |F_{Y|X}(Q_\eta[Y|X=x]|x) - F_{Y|X}(Q_\tau[Y]|x)| = f_{Y|X}(\tilde{Q}|x) |Q_\eta[Y|X=x] - Q_\tau[Y]|,$$

where \tilde{Q} is between $Q_\eta[Y|X=x]$ and $Q_\tau[Y]$. Now, $f_{Y|X}(\tilde{Q}|x) > 0$. Moreover, if $\eta > \xi_\tau(x)$, then $\eta > \xi_\tau(x) + \Delta$, so that $Q_\eta[Y|X=x] > Q_{\xi_\tau(x)+\Delta}[Y|X=x] > Q_{\xi_\tau(x)}[Y|X=x] = Q_\tau[Y]$, where we take Δ small enough such that $\eta < 1$. The same analysis can be carried out for $\eta < \xi_\tau(x)$, in

which case: $Q_\eta[Y|X = x] < Q_{\tilde{\xi}_\tau(x)-\Delta}[Y|X = x] < Q_{\tilde{\xi}_\tau(x)}[Y|X = x] = Q_\tau[Y]$. Therefore,

$$\begin{aligned} \inf_{\eta:|\eta-\tilde{\xi}_\tau(x)|>\Delta} |\Psi_\tau(\eta|X)| &= \inf_{\eta:|\eta-\tilde{\xi}_\tau(x)|>\Delta} f_{Y|X}(\tilde{Q}|x) |Q_\eta[Y|X = x] - Q_\tau[Y]| \\ &\geq \inf_{\eta:|\eta-\tilde{\xi}_\tau(x)|>\Delta} f_{Y|X}(\tilde{Q}|x) \times \inf_{\eta:|\eta-\tilde{\xi}_\tau(x)|>\Delta} |Q_\eta[Y|X = x] - Q_\tau[Y]| \\ &\geq \inf_{y \in \mathbb{R}} f_{Y|X}(y|x) \times \min \left\{ |Q_{\tilde{\xi}_\tau(x)+\Delta}[Y|X = x] - Q_\tau[Y]|, |Q_{\tilde{\xi}_\tau(x)-\Delta}[Y|X = x] - Q_\tau[Y]| \right\} \\ &> 0, \end{aligned}$$

where we have used that $f_{Y|X}(y|x)$ is bounded away from zero. Finally, we can invoke Theorem 5.9 in [van der Vaart \(1998\)](#), since we also showed that $\Psi_{\tau,n}(\hat{\xi}_\tau(x)|x) = o_p(n^{-1/2})$, therefore, $\hat{\xi}_\tau(x) \xrightarrow{P} \xi_\tau(x)$.

Having shown consistency, we now prove it is actually \sqrt{n} -consistent. To that end, we use [Assumption 1.5](#):

$$\begin{aligned} o_p(n^{-1/2}) &= x' \hat{\beta}(\hat{\xi}_\tau(x)) - \hat{Q}_Y[\tau] \\ &= x' (\hat{\beta}(\hat{\xi}_\tau(x)) - \beta(\hat{\xi}_\tau(x))) + x' (\beta(\hat{\xi}_\tau(x)) - \beta(\xi_\tau(x))) \\ &\quad + \underbrace{x' \beta(\xi_\tau(x)) - Q_Y[\tau]}_{=\Psi_\tau(\xi_\tau(x)|x)=0} - (\hat{Q}_Y[\tau] - Q_Y[\tau]). \end{aligned}$$

By [Assumptions 1.1, 1.2](#), we have that $F_{Y|X}(x' \beta(\eta)|x) = \eta$, so that $x' \beta'(\eta) = f_{Y|X}(x' \beta(\eta)|x)^{-1} > 0$, so that we can do a first order term-by-term Taylor expansion to obtain

$$\beta(\hat{\xi}_\tau(x)) - \beta(\xi_\tau(x)) = \dot{\beta}(\xi_\tau(x)) (\hat{\xi}_\tau(x) - \xi_\tau(x)) + o_p(|\hat{\xi}_\tau(x) - \xi_\tau(x)|).$$

Here, $\dot{\beta}(\xi_\tau(x))$ is the Jacobian vector: the derivative of the map $\tau \mapsto \beta(\tau)$ and $o_p(|\hat{\xi}_\tau(x) - \xi_\tau(x)|)$ is a vector of residuals of the expansion. Plugging this into the previous display, we obtain

$$\begin{aligned} o_p(n^{-1/2}) &= x' (\hat{\beta}(\hat{\xi}_\tau(x)) - \beta(\hat{\xi}_\tau(x))) + x' \dot{\beta}(\xi_\tau(x)) (\hat{\xi}_\tau(x) - \xi_\tau(x)) \\ &\quad + o_p(|\hat{\xi}_\tau(x) - \xi_\tau(x)|) - (\hat{Q}_Y[\tau] - Q_Y[\tau]). \end{aligned} \tag{A.2}$$

Here the term $o_p(|\hat{\xi}_\tau(x) - \xi_\tau(x)|)$ is scalar-valued and collects all the terms from $x' o_p(|\hat{\xi}_\tau(x) - \xi_\tau(x)|)$. Now, $\hat{Q}_Y[\tau] - Q_Y[\tau] = O_p(n^{-1/2})$ by [Assumption 1.4](#). Also, by [Assumption 1.3](#) $\hat{\beta}(\hat{\xi}_\tau(x)) - \beta(\hat{\xi}_\tau(x)) \leq \sup_{\eta \in [\epsilon, 1-\epsilon]} |\hat{\beta}(\eta) - \beta(\eta)| = O_p(n^{-1/2})$. Therefore, since $x' \dot{\beta}(\xi_\tau(x)) \neq 0$, we have that $|\hat{\xi}_\tau(x) - \xi_\tau(x)| = O_p(n^{-1/2})$.

Finally to obtain the asymptotic distribution, we go back to [\(A.2\)](#), and using the stochastic equicontinuity guaranteed by [Assumption 1.3](#), we replace $\hat{\beta}(\hat{\xi}_\tau(x)) - \beta(\hat{\xi}_\tau(x))$ by $\hat{\beta}(\xi_\tau(x)) -$

$\beta(\xi_\tau(x)) + o_p(n^{-1/2})$. Therefore, we obtain

$$\begin{aligned}\hat{\xi}_\tau(x) - \xi_\tau(x) &= -\frac{1}{x'\dot{\beta}(\xi_\tau(x))}x'(\hat{\beta}(\xi_\tau(x)) - \beta(\xi_\tau(x))) + \frac{1}{x'\dot{\beta}(\xi_\tau(x))}(\hat{Q}_Y[\tau] - Q_Y[\tau]) + o_p(n^{-1/2}) \\ &= -\frac{1}{x'\dot{\beta}(\xi_\tau(x))}\frac{1}{n}\sum_{i=1}^n x'\Psi_i(\xi_\tau(x)) + \frac{1}{x'\dot{\beta}(\xi_\tau(x))}\frac{1}{n}\sum_{i=1}^n \psi_i(\tau) + o_p(n^{-1/2}).\end{aligned}$$

To obtain the main statement of the theorem we write

$$\begin{aligned}\hat{\beta}_1(\hat{\xi}_\tau(x)) - \beta_1(\xi_\tau(x)) &= \hat{\beta}_1(\hat{\xi}_\tau(x)) - \beta_1(\hat{\xi}_\tau(x)) + \beta_1(\hat{\xi}_\tau(x)) - \beta_1(\xi_\tau(x)) \\ &= \hat{\beta}_1(\xi_\tau(x)) - \beta_1(\xi_\tau(x)) + \dot{\beta}_1(\xi_\tau(x))(\hat{\xi}_\tau(x) - \xi_\tau(x)) + o_p(n^{-1/2}).\end{aligned}$$

A.3 Proof of Theorem 2

To alleviate notation, we write:

$$\begin{aligned}\hat{m}_1(q, b, e) &:= \frac{1}{n}\sum_{i=1}^n K_h(y_i - q) \cdot b(e(x_i)) \\ \hat{m}_2(q) &:= \frac{1}{n}\sum_{i=1}^n K_h(y_i - q).\end{aligned}$$

Thus, our estimator of $UQPE_{X_1}(\tau)$ can be written as

$$\widehat{UQPE}_{X_1}(\tau) = \frac{\hat{m}_1(\hat{Q}_Y[\tau], \hat{\beta}_1, \hat{\xi}_\tau)}{\hat{m}_2(\hat{Q}_Y[\tau])}.$$

The unfeasible version is then

$$\widetilde{UQPE}_{X_1}(\tau) = \frac{\hat{m}_1(Q_Y[\tau], \beta_1, \xi_\tau)}{\hat{m}_2(Q_Y[\tau])}.$$

Consider the difference

$$\begin{aligned}\widehat{UQPE}_{X_1}(\tau) - \widetilde{UQPE}_{X_1}(\tau) &= \frac{\hat{m}_1(\hat{Q}_Y[\tau], \hat{\beta}_1, \hat{\xi}_\tau)}{\hat{m}_2(\hat{Q}_Y[\tau])} - \frac{\hat{m}_1(Q_Y[\tau], \beta_1, \xi_\tau)}{\hat{m}_2(Q_Y[\tau])} \\ &= \frac{\hat{m}_2(Q_Y[\tau])\hat{m}_1(\hat{Q}_Y[\tau], \hat{\beta}_1, \hat{\xi}_\tau) - \hat{m}_2(\hat{Q}_Y[\tau])\hat{m}_1(Q_Y[\tau], \beta_1, \xi_\tau)}{\hat{m}_2(\hat{Q}_Y[\tau])\hat{m}_2(Q_Y[\tau])} \\ &= \frac{\hat{m}_1(\hat{Q}_Y[\tau], \hat{\beta}_1, \hat{\xi}_\tau) - \hat{m}_1(Q_Y[\tau], \beta_1, \xi_\tau)}{\hat{m}_2(\hat{Q}_Y[\tau])} \\ &\quad - \frac{\hat{m}_1(Q_Y[\tau], \beta_1, \xi_\tau)}{\hat{m}_2(\hat{Q}_Y[\tau])\hat{m}_2(Q_Y[\tau])}(\hat{m}_2(\hat{Q}_Y[\tau]) - \hat{m}_2(Q_Y[\tau])).\end{aligned}\tag{A.3}$$

First we focus on the second term of (A.3). We note that

$$\hat{m}_2(\hat{Q}_Y[\tau]) := \frac{1}{n} \sum_{i=1}^n K_h(y_i - \hat{Q}_Y[\tau]) = \hat{f}_Y(\hat{Q}_Y[\tau])$$

is an estimator of the density of Y evaluated at $\hat{Q}_Y[\tau]$, the estimator of $Q_Y[\tau]$; while

$$\hat{m}_2(Q_Y[\tau]) := \frac{1}{n} \sum_{i=1}^n K_h(y_i - Q_Y[\tau]) = \hat{f}_Y(Q_Y[\tau])$$

is an estimator of the density of Y evaluated at $Q_Y[\tau]$. We now show that

$$\begin{aligned} \hat{m}_2(\hat{Q}_Y[\tau]) - \hat{m}_2(Q_Y[\tau]) &= \hat{f}_Y(\hat{Q}_Y[\tau]) - \hat{f}_Y(Q_Y[\tau]) \\ &= f'_Y(Q_Y[\tau])(\hat{Q}_Y[\tau] - Q_Y[\tau]) + o_p(n^{-1/2}h^{-1/2}), \end{aligned} \quad (\text{A.4})$$

which implies that $\hat{m}_2(\hat{Q}_Y[\tau]) - \hat{m}_2(Q_Y[\tau]) = o_p(n^{-1/2}h^{-1/2})$.

By Taylor's theorem, we have that for some $\tilde{Q}_Y[\tau]$ between $\hat{Q}_Y[\tau]$ and $Q_Y[\tau]$

$$\begin{aligned} \hat{f}_Y(\hat{Q}_Y[\tau]) - \hat{f}_Y(Q_Y[\tau]) &= \hat{f}'_Y(Q_Y[\tau])(\hat{Q}_Y[\tau] - Q_Y[\tau]) + \frac{1}{2} \hat{f}''_Y(\tilde{Q}_Y[\tau])(\hat{Q}_Y[\tau] - Q_Y[\tau])^2 \\ &= f'_Y(Q_Y[\tau])(\hat{Q}_Y[\tau] - Q_Y[\tau]) \\ &\quad + (\hat{f}'_Y(Q_Y[\tau]) - f'_Y(Q_Y[\tau]))(\hat{Q}_Y[\tau] - Q_Y[\tau]) \\ &\quad + \frac{1}{2} \hat{f}''_Y(\tilde{Q}_Y[\tau])(\hat{Q}_Y[\tau] - Q_Y[\tau])^2. \end{aligned}$$

We need to show that the second and third terms are $o_p(n^{-1/2}h^{-1/2})$. We start with the third term. The second derivative of the kernel estimator is

$$\hat{f}''(q) = \frac{1}{nh^3} \sum_{i=1}^n K''\left(\frac{y_i - q}{h}\right).$$

Since the second derivative K'' is bounded by Assumption 2: $\sup_u |K''(u)| < \infty$, then

$$\hat{f}''(\tilde{Q}_Y[\tau]) \leq \frac{1}{nh^3} \sum_{i=1}^n \sup_u |K''(u)| = \frac{1}{h^3} \sup_u |K''(u)|,$$

therefore

$$\hat{f}''(\tilde{Q}_Y[\tau]) = O_p(h^{-3}).$$

This means that

$$\frac{1}{2} \hat{f}''(\tilde{Q}_Y[\tau])(\hat{Q}_Y[\tau] - Q_Y[\tau])^2 = O_p(n^{-1}h^{-3}).$$

Now,

$$n^{1/2}h^{1/2}\frac{1}{2}\hat{f}''(\tilde{Q}_Y[\tau])(\hat{Q}_Y[\tau] - Q_Y[\tau])^2 = n^{1/2}h^{1/2}O_p(n^{-1}h^{-3}) = O_p(n^{-1/2}h^{-5/2}) = o_p(1),$$

because $nh^5 \rightarrow \infty$ by Assumption 4. Therefore,

$$\frac{1}{2}\hat{f}''(\tilde{Q}_Y[\tau])(\hat{Q}_Y[\tau] - Q_Y[\tau])^2 = o_p(n^{-1/2}h^{-1/2}).$$

Now, for the other term, $(\hat{f}'_Y(Q_Y[\tau]) - f'_Y(Q_Y[\tau]))(\hat{Q}_Y[\tau] - Q_Y[\tau])$, we focus on the derivative. The first derivative of the kernel estimator is

$$\hat{f}'(q) = -\frac{1}{nh^2} \sum_{i=1}^n K' \left(\frac{y_i - q}{h} \right).$$

We approximate its mean and variance under Assumption 2. We will use the fact that, by Assumption 2, $\int K'(u)du = 0$, $\int K'(u)udu = -1$, $\int K'(u)u^j du = 0$ for $j = 2, \dots, r$, and $\int K'(u)u^{r+1}du < \infty$. In the following, $f_Y^{(j)}$ denotes the j -th derivative.

$$\begin{aligned} E[\hat{f}'(q)] &= -\frac{1}{h^2} E \left[K' \left(\frac{Y - q}{h} \right) \right] \\ &= -\frac{1}{h^2} \int_{\mathbb{R}} K' \left(\frac{y - q}{h} \right) f_Y(y) dy \\ &= -\frac{1}{h} \int_{\mathbb{R}} K'(u) f_Y(q + hu) du \\ &= -\frac{1}{h} \int_{\mathbb{R}} K'(u) \left[f_Y(q) + \sum_{j=1}^r \frac{h^j u^j f_Y^{(j)}(q)}{j!} + \frac{h^{r+1} u^{r+1} f_Y^{(r+1)}(\tilde{q})}{(r+1)!} \right] du \\ &= f'_Y(q) - \frac{h^r}{(r+1)!} \int_{\mathbb{R}} K'(u) u^{r+1} f_Y^{(r+1)}(\tilde{q}) du. \end{aligned}$$

Since, by Assumption 3, the derivatives are bounded, the bias is

$$E[\hat{f}'(q)] - f'_Y(q) = O(h^r).$$

For the variance term, we have

$$\begin{aligned}
\text{Var}[\hat{f}'(q)] &= \frac{1}{nh^4} \text{Var} \left[K' \left(\frac{Y-q}{h} \right) \right] \\
&= \frac{1}{nh^4} E \left[K' \left(\frac{Y-q}{h} \right)^2 \right] - \frac{1}{nh^4} E \left[K' \left(\frac{Y-q}{h} \right) \right]^2 \\
&= \frac{1}{nh^4} \int_{\mathbb{R}} K' \left(\frac{y-q}{h} \right)^2 f_Y(y) dy - \frac{1}{nh^4} \left[\int_{\mathbb{R}} K' \left(\frac{y-q}{h} \right) f_Y(y) dy \right]^2 \\
&= \frac{1}{nh^3} \int_{\mathbb{R}} K'(u)^2 f_Y(q+hu) du - \frac{1}{nh^2} \left[\int_{\mathbb{R}} K'(u) f_Y(q+hu) du \right]^2 \\
&= \frac{1}{nh^3} \int_{\mathbb{R}} K'(u)^2 [f_Y(q) + hu f_Y'(q)] du - O(n^{-1}h^{-2}) \\
&= \frac{f_Y(q)}{nh^3} \int_{\mathbb{R}} K'(u)^2 du + o(n^{-1}h^{-2}) - O(n^{-1}h^{-2}).
\end{aligned}$$

Therefore,

$$\hat{f}'(q) - f'(q) = O_p(n^{-1/2}h^{-3/2} + h^r).$$

Finally, we have

$$\begin{aligned}
n^{1/2}h^{1/2}(\hat{f}'_Y(Q_Y[\tau]) - f'_Y(Q_Y[\tau]))(\hat{Q}_Y[\tau] - Q_Y[\tau]) &= n^{1/2}h^{1/2}O_p(n^{-1/2}h^{-3/2} + h^r)O_p(n^{-1/2}) \\
&= O_p(n^{-1/2}h^{-1} + h^{r+1/2})O_p(1) \\
&= o_p(1),
\end{aligned}$$

since $n^{1/2}h = (nh^2)^{1/2} \rightarrow \infty$ since $nh^2 \rightarrow \infty$ because by Assumption 4 $nh^5 \rightarrow \infty$. Putting all the results together, we obtain

$$\hat{f}_Y(\hat{Q}_Y[\tau]) - \hat{f}_Y(Q_Y[\tau]) = f'_Y(Q_Y[\tau])(\hat{Q}_Y[\tau] - Q_Y[\tau]) + o_p(n^{-1/2}h^{-1/2}).$$

Thus, we obtain that

$$\hat{m}_2(\hat{Q}_Y[\tau]) - \hat{m}_2(Q_Y[\tau]) = o_p(n^{-1/2}h^{-1/2}). \quad (\text{A.5})$$

Now we focus on the first term of (A.3). For the numerator, consider the following decompo-

sition

$$\begin{aligned}
\hat{m}_1(\hat{Q}_Y[\tau], \hat{\beta}_1, \hat{\xi}_\tau) - \hat{m}_1(Q_Y[\tau], \beta_1, \xi_\tau) &= \frac{1}{n} \sum_{i=1}^n K_h(y_i - \hat{Q}_Y[\tau]) \cdot \hat{\beta}_1(\hat{\xi}_\tau(x_i)) - \frac{1}{n} \sum_{i=1}^n K_h(y_i - Q_Y[\tau]) \cdot \beta_1(\xi_\tau(x_i)) \\
&= \frac{1}{n} \sum_{i=1}^n \underbrace{[K_h(y_i - \hat{Q}_Y[\tau]) - K_h(y_i - Q_Y[\tau])]}_{:=T_1} \cdot \beta_1(\xi_\tau(x_i)) \\
&\quad + \frac{1}{n} \sum_{i=1}^n \underbrace{K_h(y_i - Q_Y[\tau]) \cdot [\hat{\beta}_1(\hat{\xi}_\tau(x_i)) - \beta_1(\xi_\tau(x_i))]}_{:=T_2} \\
&\quad + \frac{1}{n} \sum_{i=1}^n \underbrace{[K_h(y_i - \hat{Q}_Y[\tau]) - K_h(y_i - Q_Y[\tau])] \cdot [\hat{\beta}_1(\hat{\xi}_\tau(x_i)) - \beta_1(\xi_\tau(x_i))]}_{:=T_3} \\
&= T_1 + T_2 + T_3. \tag{A.6}
\end{aligned}$$

We start with T_1 and we do a second order Taylor expansion:

$$\begin{aligned}
T_1 &:= \frac{1}{n} \sum_{i=1}^n [K_h(y_i - \hat{Q}_Y[\tau]) - K_h(y_i - Q_Y[\tau])] \cdot \beta_1(\xi_\tau(x_i)) \\
&= (\hat{Q}_Y[\tau] - Q_Y[\tau]) \frac{1}{n} \sum_{i=1}^n \left. \frac{\partial K_h(y_i - q)}{\partial q} \right|_{q=Q_Y[\tau]} \cdot \beta_1(\xi_\tau(x_i)) \\
&\quad + \frac{1}{2} (\hat{Q}_Y[\tau] - Q_Y[\tau])^2 \frac{1}{n} \sum_{i=1}^n \left. \frac{\partial^2 K_h(y_i - q)}{\partial^2 q} \right|_{q=\bar{q}} \cdot \beta_1(\xi_\tau(x_i)). \tag{A.7}
\end{aligned}$$

Consider the first term. Let $f_{Y,X}^{(j)}(y, x)$ denote the j -th partial derivative of $f_{Y,X}(y, x)$ with respect to y . The expected value is

$$\begin{aligned}
E \left[\frac{1}{n} \sum_{i=1}^n \left. \frac{\partial K_h(Y_i - q)}{\partial q} \right|_{q=Q_Y[\tau]} \cdot \beta_1(\xi_\tau(X_i)) \right] &= E \left[\left. \frac{\partial K_h(Y - q)}{\partial q} \right|_{q=Q_Y[\tau]} \cdot \beta_1(\xi_\tau(X)) \right] \\
&= -\frac{1}{h^2} E \left[K' \left(\frac{Y - Q_Y[\tau]}{h} \right) \cdot \beta_1(\xi_\tau(X)) \right] \\
&= -\frac{1}{h^2} \int \int K' \left(\frac{y - Q_Y[\tau]}{h} \right) \beta_1(\xi_\tau(x)) f_{Y,X}(y, x) dy dx \\
&= -\frac{1}{h} \int \int K'(u) \beta_1(\xi_\tau(x)) f_{Y,X}(Q_Y[\tau] + hu, x) dudx \\
&= -\frac{1}{h} \int \int K'(u) \beta_1(\xi_\tau(x)) \left[f_{Y,X}(Q_Y[\tau], x) \right. \\
&\quad \left. + \sum_{j=1}^r \frac{h^j u^j f_{Y,X}^{(j)}(Q_Y[\tau], x)}{j!} + \frac{h^{r+1} u^{r+1} f_{Y,X}^{(r+1)}(\tilde{Q}_Y[\tau], x)}{(r+1)!} \right] dudx,
\end{aligned}$$

where we used Assumption 3 to expand the joint density. The properties of the kernel of As-

sumption 2 yield

$$\begin{aligned}
E \left[\frac{1}{n} \sum_{i=1}^n \frac{\partial K_h(Y_i - q)}{\partial q} \Big|_{q=Q_Y[\tau]} \cdot \beta_1(\xi_\tau(X_i)) \right] &= -\frac{1}{h} \int \int K'(u) \beta_1(\xi_\tau(x)) \left[f_{Y,X}(Q_Y[\tau], x) \right. \\
&\quad \left. + \sum_{j=1}^r \frac{h^j u^j f_{Y,X}^{(j)}(Q_Y[\tau], x)}{j!} + \frac{h^{r+1} u^{r+1} f_{Y,X}^{(r+1)}(\tilde{Q}_Y[\tau], x)}{(r+1)!} \right] dudx \\
&= \int \beta_1(\xi_\tau(x)) f_{Y,X}^{(1)}(Q_Y[\tau], x) dx \\
&\quad - h^r \int \int K'(u) \beta_1(\xi_\tau(x)) u^{r+1} f_{Y,X}^{(r+1)}(\tilde{Q}_Y[\tau], x) \beta_1(\xi_\tau(x)) dx du.
\end{aligned}$$

Therefore, the bias is of order $O(h^r)$:

$$E \left[\frac{1}{n} \sum_{i=1}^n \frac{\partial K_h(Y_i - q)}{\partial q} \Big|_{q=Q_Y[\tau]} \cdot \beta_1(\xi_\tau(X_i)) \right] = \int \beta_1(\xi_\tau(x)) f_{Y,X}^{(1)}(Q_Y[\tau], x) dx + O(h^r).$$

For the variance, we have

$$\begin{aligned}
\text{Var} \left[\frac{1}{n} \sum_{i=1}^n \frac{\partial K_h(Y_i - q)}{\partial q} \Big|_{q=Q_Y[\tau]} \cdot \beta_1(\xi_\tau(X_i)) \right] &= \frac{1}{nh^4} \text{Var} \left[K' \left(\frac{Y - Q_Y[\tau]}{h} \right) \beta_1(\xi_\tau(X)) \right] \\
&= \frac{1}{nh^4} E \left[K' \left(\frac{Y - Q_Y[\tau]}{h} \right)^2 \beta_1(\xi_\tau(X))^2 \right] \\
&\quad - \frac{1}{nh^4} E \left[K' \left(\frac{Y - Q_Y[\tau]}{h} \right) \beta_1(\xi_\tau(X)) \right]^2.
\end{aligned}$$

We take care of each term at a time.

$$\begin{aligned}
\frac{1}{nh^4} E \left[K' \left(\frac{Y - Q_Y[\tau]}{h} \right)^2 \beta_1(\xi_\tau(X))^2 \right] &= \frac{1}{nh^4} \int \int K' \left(\frac{y - q}{h} \right)^2 \beta_1(\xi_\tau(x))^2 f_{Y,X}(Q_Y[\tau], x) dy dx \\
&= \frac{1}{nh^3} \int \int K'(u)^2 \beta_1(\xi_\tau(x))^2 f_{Y,X}(Q_Y[\tau] + hu, x) dudx \\
&= \frac{1}{nh^3} \int \int K'(u)^2 \beta_1(\xi_\tau(x))^2 \left[f_{Y,X}(Q_Y[\tau], x) + hu f_{Y,X}^{(1)}(\tilde{Q}_Y[\tau], x) \right] dudx \\
&= \frac{1}{nh^3} \int \int K'(u)^2 \beta_1(\xi_\tau(x))^2 f_{Y,X}(Q_Y[\tau], x) dudx + o(n^{-1}h^{-2}).
\end{aligned}$$

For the other term, we have

$$\begin{aligned}
\frac{1}{nh^4} E \left[K' \left(\frac{Y - Q_Y[\tau]}{h} \right) \beta_1(\xi_\tau(X)) \right]^2 &= \frac{1}{nh^4} \left[\int \int K' \left(\frac{y - Q_Y[\tau]}{h} \right) \beta_1(\xi_\tau(x)) f_{Y,X}(y, x) dy dx \right]^2 \\
&= \frac{1}{nh^2} \left[\int \int K'(u) \beta_1(\xi_\tau(x)) f_{Y,X}(Q_Y[\tau] + hu, x) dudx \right]^2 \\
&= O(n^{-1}h^{-2}).
\end{aligned}$$

Combining the bias and variance results, we obtain

$$E \left[\frac{1}{n} \sum_{i=1}^n \frac{\partial K_h(Y_i - q)}{\partial q} \Big|_{q=Q_Y[\tau]} \cdot \beta_1(\xi_\tau(X_i)) \right] - \int \beta_1(\xi_\tau(x)) f_{Y,X}^{(1)}(Q_Y[\tau], x) dx = O_p(n^{-1/2}h^{-3/2} + h^r).$$

For the remaining term in (A.7) we use the fact that by Assumption 2, the second derivative of the kernel is bounded:

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \frac{\partial^2 K_h(y_i - q)}{\partial^2 q} \Big|_{q=\bar{q}} \cdot \beta_1(\xi_\tau(x_i)) &= \frac{1}{nh^3} \sum_{i=1}^n K'' \left(\frac{y_i - \bar{q}}{h} \right) \cdot \beta_1(\xi_\tau(x_i)) \\ &\leq \frac{\sup_{u \in \mathbb{R}} |K''(u)|}{h^3} \cdot \frac{1}{n} \sum_{i=1}^n \beta_1(\xi_\tau(x_i)) \\ &= O_p(h^{-3}), \end{aligned}$$

since $\frac{1}{n} \sum_{i=1}^n \beta_1(\xi_\tau(x_i)) = O_p(1)$.

Now we show that T_2 in (A.6) satisfies

$$T_2 := \frac{1}{n} \sum_{i=1}^n K_h(y_i - Q_Y[\tau]) \cdot [\hat{\beta}_1(\hat{\xi}_\tau(x_i)) - \beta_1(\xi_\tau(x_i))] = o_p(n^{-1/2}h^{-1/2}).$$

We use the following decomposition, similar to the one in Theorem 1:

$$\hat{\beta}_1(\hat{\xi}_\tau(x)) - \beta_1(\xi_\tau(x)) = \hat{\beta}_1(\hat{\xi}_\tau(x)) - \beta_1(\hat{\xi}_\tau(x)) + \beta_1(\hat{\xi}_\tau(x)) - \beta_1(\xi_\tau(x)).$$

We have

$$\begin{aligned} T_2 &:= \frac{1}{n} \sum_{i=1}^n K_h(y_i - Q_Y[\tau]) \cdot [\hat{\beta}_1(\hat{\xi}_\tau(x_i)) - \beta_1(\xi_\tau(x_i))] \\ &= \frac{1}{n} \sum_{i=1}^n K_h(y_i - Q_Y[\tau]) \cdot [\hat{\beta}_1(\hat{\xi}_\tau(x_i)) - \beta_1(\hat{\xi}_\tau(x_i))] + \frac{1}{n} \sum_{i=1}^n K_h(y_i - Q_Y[\tau]) \cdot [\beta_1(\hat{\xi}_\tau(x_i)) - \beta_1(\xi_\tau(x_i))] \\ &\leq \sup_{\eta \in [\epsilon, 1-\epsilon]} |\hat{\beta}(\eta) - \beta(\eta)| \frac{1}{n} \sum_{i=1}^n K_h(y_i - Q_Y[\tau]) + \frac{1}{n} \sum_{i=1}^n K_h(y_i - Q_Y[\tau]) \cdot [\beta_1(\hat{\xi}_\tau(x_i)) - \beta_1(\xi_\tau(x_i))]. \end{aligned}$$

Here we use $\sup_{\eta \in [\epsilon, 1-\epsilon]} |\hat{\beta}(\eta) - \beta(\eta)| = O_p(n^{-1/2})$ and $\frac{1}{n} \sum_{i=1}^n K_h(y_i - Q_Y[\tau]) = O_p(1)$, to conclude that the first term is $o_p(n^{-1/2}h^{-1/2})$. By the Cauchy-Schwarz inequality, the second term is bounded by

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n K_h(y_i - Q_Y[\tau]) \cdot [\beta_1(\hat{\xi}_\tau(x_i)) - \beta_1(\xi_\tau(x_i))] &\leq \\ \frac{1}{\sqrt{h}} \left[\frac{1}{nh} \sum_{i=1}^n K \left(\frac{y_i - Q_Y[\tau]}{h} \right)^2 \right]^{1/2} \cdot \left[\frac{1}{n} \sum_{i=1}^n [\beta_1(\hat{\xi}_\tau(x_i)) - \beta_1(\xi_\tau(x_i))]^2 \right]^{1/2}. \end{aligned}$$

Here

$$\frac{1}{nh} \sum_{i=1}^n K \left(\frac{y_i - Q_Y[\tau]}{h} \right)^2 = O_p(1).$$

For the other term, we use Markov's inequality

$$\begin{aligned} \Pr \left[\frac{1}{n} \sum_{i=1}^n [\beta_1(e(X_i)) - \beta_1(\xi_\tau(X_i))]^2 > \varepsilon \right] \Big|_{e=\hat{\xi}_\tau} &\leq \frac{1}{n\varepsilon} E \left[\sum_{i=1}^n [\beta_1(e(X_i)) - \beta_1(\xi_\tau(X_i))]^2 \right] \Big|_{e=\hat{\xi}_\tau} \\ &\leq \frac{1}{\varepsilon} E \left[[\beta_1(e(X_i)) - \beta_1(\xi_\tau(X_i))]^2 \right] \Big|_{e=\hat{\xi}_\tau}. \end{aligned}$$

Since by Assumption 5, $E \left[(n^{1/4} [\beta_1(e(X_i)) - \beta_1(\xi_\tau(X_i))])^2 \right] \Big|_{e=\hat{\xi}_\tau} = o_p(1)$, then

$$T_2 := \frac{1}{n} \sum_{i=1}^n K_h(y_i - Q_Y[\tau]) \cdot [\beta_1(\hat{\xi}_\tau(x_i)) - \beta_1(\xi_\tau(x_i))] = o_p(n^{-1/2}).$$

Hence,

$$T_2 := \frac{1}{n} \sum_{i=1}^n K_h(y_i - Q_Y[\tau]) \cdot [\beta_1(\hat{\xi}_\tau(x_i)) - \beta_1(\xi_\tau(x_i))] = o_p(n^{-1/2}h^{-1/2}).$$

For T_3 in (A.6), we have

$$\begin{aligned} T_3 &:= \frac{1}{n} \sum_{i=1}^n [K_h(y_i - \hat{Q}_Y[\tau]) - K_h(y_i - Q_Y[\tau])] \cdot [\hat{\beta}_1(\hat{\xi}_\tau(x_i)) - \beta_1(\xi_\tau(x_i))] \\ &\leq \sup_{\eta \in [\varepsilon, 1-\varepsilon]} |\hat{\beta}(\eta) - \beta(\eta)| \frac{1}{n} \sum_{i=1}^n [K_h(y_i - \hat{Q}_Y[\tau]) - K_h(y_i - Q_Y[\tau])] \\ &\quad + \frac{1}{n} \sum_{i=1}^n [K_h(y_i - \hat{Q}_Y[\tau]) - K_h(y_i - Q_Y[\tau])] \cdot [\beta_1(\hat{\xi}_\tau(x_i)) - \beta_1(\xi_\tau(x_i))]. \end{aligned}$$

By (A.5) we have that

$$\begin{aligned} \sup_{\eta \in [\varepsilon, 1-\varepsilon]} |\hat{\beta}(\eta) - \beta(\eta)| \frac{1}{n} \sum_{i=1}^n [K_h(y_i - \hat{Q}_Y[\tau]) - K_h(y_i - Q_Y[\tau])] &= O_p(n^{-1/2}) o_p(n^{-1/2}h^{-1/2}) \\ &= o_p(n^{-1}h^{-1/2}). \end{aligned}$$

For other term we have

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n [K_h(y_i - \hat{Q}_Y[\tau]) - K_h(y_i - Q_Y[\tau])] \cdot [\beta_1(\hat{\xi}_\tau(x_i)) - \beta_1(\xi_\tau(x_i))] \\
&= (\hat{Q}_Y[\tau] - Q_Y[\tau]) \frac{1}{nh^2} \sum_{i=1}^n K' \left(\frac{y_i - \tilde{Q}_Y[\tau]}{h} \right) \cdot [\beta_1(\hat{\xi}_\tau(x_i)) - \beta_1(\xi_\tau(x_i))] \\
&\leq (\hat{Q}_Y[\tau] - Q_Y[\tau]) \frac{1}{h} \left[\frac{1}{nh^2} \sum_{i=1}^n K' \left(\frac{y_i - \tilde{Q}_Y[\tau]}{h} \right)^2 \right]^{1/2} \cdot \left[\frac{1}{n} \sum_{i=1}^n [\beta_1(\hat{\xi}_\tau(x_i)) - \beta_1(\xi_\tau(x_i))]^2 \right]^{1/2} \\
&= O_p(n^{-1/2}) \frac{1}{h} o_p(n^{-1/4}) \\
&= o_p(n^{-3/4}h^{-1}),
\end{aligned}$$

since

$$\frac{1}{nh^2} \sum_{i=1}^n K' \left(\frac{y_i - \tilde{Q}_Y[\tau]}{h} \right)^2 = O_p(1).$$

Now, $n^{1/2}h^{1/2}n^{-3/4}h^{-1} = n^{-1/4}h^{-1/2}$, so we need $nh^2 \rightarrow \infty$, which is satisfied by Assumption 4 which states $nh^5 \rightarrow \infty$. Therefore, T_3 is $o_p(n^{-1/2}h^{-1/2})$. This means that

$$\hat{m}_1(\hat{Q}_Y[\tau], \hat{\beta}_1, \hat{\xi}_\tau) - \hat{m}_1(Q_Y[\tau], \beta_1, \xi_\tau) = o_p(n^{-1/2}h^{-1/2}). \tag{A.8}$$

Therefore, (A.5) and (A.8) imply that (A.3) is actually

$$\widehat{UQPPE}_{X_1}(\tau) - \widetilde{UQPPE}_{X_1}(\tau) = o_p(n^{-1/2}h^{-1/2}).$$

A.4 Proof of Corollary 1

Recall that by equation (11), we have that

$$UQPPE_{X_1}(\tau) = E[\beta_1(\xi_\tau(X)) | Y = Q_Y[\tau]],$$

and that we defined

$$U_\tau := \beta_1(\xi_\tau(X)) - E[\beta_1(\xi_\tau(X)) | Y]$$

and, by construction, $E[U_\tau | Y] = 0$ a.s. and, by Assumption 6, $E[U_\tau^2 | Y = y] = \sigma_\tau^2(y) < \infty$ for every y in the support of Y . Moreover, for (x_i, y_i) , we have

$$u_{\tau,i} := \beta_1(\xi_\tau(x_i)) - E[\beta_1(\xi_\tau(X)) | Y = y_i]. \tag{A.9}$$

We focus on

$$\begin{aligned}
\sqrt{nh} \left(\widetilde{UQPPE}_{X_1}(\tau) - UQPPE_{X_1}(\tau) \right) &= \sqrt{nh} \left(\frac{\sum_{i=1}^n K_h(y_i - Q_Y[\tau]) \cdot \beta_1(\xi_\tau(x_i))}{\sum_{i=1}^n K_h(y_i - Q_Y[\tau])} - UQPPE_{X_1}(\tau) \right) \\
&= \sqrt{nh} \left(\frac{\sum_{i=1}^n K_h(y_i - Q_Y[\tau]) \cdot [\beta_1(\xi_\tau(x_i)) - UQPPE_{X_1}(\tau)]}{\sum_{i=1}^n K_h(y_i - Q_Y[\tau])} \right) \\
&= \sqrt{nh} \left(\frac{\sum_{i=1}^n K_h(y_i - Q_Y[\tau]) \cdot u_{\tau,i}}{\sum_{i=1}^n K_h(y_i - Q_Y[\tau])} \right) \\
&+ \sqrt{nh} \left(\frac{\frac{1}{n} \sum_{i=1}^n K_h(y_i - Q_Y[\tau]) \cdot [E[\beta_1(\xi_\tau(X)) | Y = y_i] - UQPPE_{X_1}(\tau)]}{\frac{1}{n} \sum_{i=1}^n K_h(y_i - Q_Y[\tau])} \right).
\end{aligned} \tag{A.10}$$

Consider the first term:

$$\sqrt{nh} \left(\frac{\sum_{i=1}^n K_h(y_i - Q_Y[\tau]) \cdot u_{\tau,i}}{\sum_{i=1}^n K_h(y_i - Q_Y[\tau])} \right) = \frac{1}{\hat{f}_Y(Q_Y[\tau])} \sum_{i=1}^n \frac{1}{\sqrt{nh}} K \left(\frac{y_i - Q_Y[\tau]}{h} \right) \cdot u_{\tau,i}.$$

For this term we use the Lindberg-Feller CLT. First we compute the variance of the sum.

$$\begin{aligned}
\text{Var} \left[\sum_{i=1}^n \frac{1}{\sqrt{nh}} K \left(\frac{Y_i - Q_Y[\tau]}{h} \right) \cdot u_{\tau,i} \right] &= \frac{1}{h} E \left[K \left(\frac{Y - Q_Y[\tau]}{h} \right)^2 \sigma_\tau^2(Y) \right] \\
&= \frac{1}{h} \int K \left(\frac{y - Q_Y[\tau]}{h} \right)^2 \sigma_\tau^2(y) f_Y(y) dy \\
&= \int K(u)^2 \sigma^2(Q_Y[\tau] + hu) f_Y(Q_Y[\tau] + hu) du \\
&\rightarrow \sigma_\tau^2(Q_Y[\tau]) f_Y(Q_Y[\tau]) \int K(u)^2 du.
\end{aligned}$$

because $\sigma_\tau^2(y)$ and $f_Y(y)$ are continuous, and are bounded. The conclusion follows from the dominated convergence theorem. We write

$$\sigma_{\tau,n}^2 := \text{Var} \left[\sum_{i=1}^n \frac{1}{\sqrt{nh}} K \left(\frac{Y_i - Q_Y[\tau]}{h} \right) \cdot u_{\tau,i} \right] \rightarrow \sigma_{\tau,0}^2.$$

To apply the Lindberg-Feller CLT, we define

$$\omega_{in} := \frac{1}{\sqrt{nh}} K \left(\frac{Y_i - Q_Y[\tau]}{h} \right).$$

We need to show that, for some $\delta > 0$,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n E \left| \frac{\omega_{in} U_{\tau,i}}{\sigma_{\tau,n}} \right|^{2+\delta} = 0.$$

We have

$$\sum_{i=1}^n E \left| \frac{\omega_{in} U_{\tau,i}}{\sigma_{\tau,n}} \right|^{2+\delta} = \sum_{i=1}^n \left| \frac{\sigma_{\tau,0}}{\sigma_{\tau,n}} \right|^{2+\delta} E \left| \frac{\omega_{in} u_{\tau,i}}{\sigma_{\tau,0}} \right|^{2+\delta}.$$

It will be sufficient to focus on

$$\begin{aligned} \sum_{i=1}^n E \left| \frac{\omega_{in} U_{\tau,i}}{\sigma_{0,\tau}} \right|^{2+\delta} &= \frac{n}{|\sigma_{0,\tau}|^{2+\delta}} E |\omega_{in} U_{\tau,i}|^{2+\delta} \\ &= \frac{n}{|\sigma_{0,\tau}|^{2+\delta}} E \left[|\omega_{in}|^{2+\delta} E[|U_{\tau,i}|^{2+\delta} | Y_i] \right] \\ &\leq C \frac{n}{|\sigma_{0,\tau}|^{2+\delta}} E \left[|\omega_{in}|^{2+\delta} \right] \\ &= \frac{Cn}{|\sigma_{0,\tau}|^{2+\delta}} (nh)^{-1-\delta/2} \int K \left(\frac{y - Q_Y[\tau]}{h} \right)^{2+\delta} f_Y(y) dy \\ &= \frac{Cn}{|\sigma_{0,\tau}|^{2+\delta}} (nh)^{-1-\delta/2} h \int K(u)^{2+\delta} f_Y(Q_Y[\tau] + hu) du \\ &= \frac{c}{(nh)^{\delta/2}} \int K(u)^{2+\delta} f_Y(Q_Y[\tau] + hu) du, \end{aligned}$$

which goes to 0 since $(nh)^{1/2} \rightarrow \infty$ and $f_Y(y)$ is continuous at $y = Q_Y[\tau]$. We have used the bound of the higher order conditional expectation of U_τ : $E[|U_{\tau,i}|^{2+\delta} | Y_i] < C$ a.s., and that $\int |K(u)|^{2+\delta} du < \infty$. Therefore,

$$\frac{1}{\hat{f}_Y(Q_Y[\tau])} \sum_{i=1}^n \frac{1}{\sqrt{nh}} K \left(\frac{y_i - Q_Y[\tau]}{h} \right) \cdot u_{\tau,i} \xrightarrow{d} N \left(0, \sigma_\tau^2(Q_Y[\tau]) f_Y(Q_Y[\tau])^{-1} \int K(u)^2 du \right) \quad (\text{A.11})$$

since $\hat{f}_Y(Q_Y[\tau]) = f_Y(Q_Y[\tau]) + o_p(1)$.

The second term in the expansion of $\sqrt{nh} \left(U\widetilde{QPE}_{X_1}(\tau) - UQPE_{X_1}(\tau) \right)$ is a bias term. We now find its rate of convergence. We start with the numerator.

$$\begin{aligned} &E \left[\frac{1}{n} \sum_{i=1}^n K_h(Y_i - Q_Y[\tau]) \cdot [E[\beta_1(\xi_\tau(X)) | Y = Y_i] - UQPE_{X_1}(\tau)] \right] \\ &= \frac{1}{h} \int_{\mathcal{Y}} K \left(\frac{y - Q_Y[\tau]}{h} \right) [E[\beta_1(\xi_\tau(X)) | Y = y] - UQPE_{X_1}(\tau)] f_Y(y) dy \\ &= \int K(u) [E[\beta_1(\xi_\tau(X)) | Y = Q_Y[\tau] + hu] - UQPE_{X_1}(\tau)] f_Y(Q_Y[\tau] + hu) du \\ &= \int K(u) [E[\beta_1(\xi_\tau(X)) | Y = Q_Y[\tau] + hu] - UQPE_{X_1}(\tau)] f_Y(Q_Y[\tau] + hu) du \\ &= \int K(u) E[\beta_1(\xi_\tau(X)) | Y = Q_Y[\tau] + hu] f_Y(Q_Y[\tau] + hu) du \\ &\quad - UQPE_{X_1}(\tau) \int_{\mathcal{Y}} K(u) f_Y(Q_Y[\tau] + hu) du. \end{aligned}$$

We do a Taylor expansion on the density and the conditional expectation and we use the fact that $\int K(u)du = 1$, $\int u^j K(u)du = 0$ when $j = 1, \dots, r-1$, and $\int u^r K(u)du < \infty$. Let $(Ef_Y)^{(j)}(q)$ denote the j -derivative with respect to y of the product $E[\beta_1(\xi_\tau(X))|Y = y] f_Y(y)$ evaluated at $y = q$. The first term is

$$\begin{aligned} & \int K(u)E[\beta_1(\xi_\tau(X))|Y = Q_Y[\tau] + hu] f_Y(Q_Y[\tau] + hu)du \\ &= UQPE_{X_1}(\tau) + \int K(u) \sum_{j=1}^{r-1} \frac{h^j u^j (Ef_Y)^{(j)}(Q_Y[\tau])}{j!} du \\ &+ \frac{h^r}{r!} \int K(u) u^r (Ef_Y)^{(r)}(\tilde{Q}_Y[\tau]) du \\ &= UQPE_{X_1}(\tau) + O(h^r), \end{aligned}$$

since the derivatives are uniformly bounded. Now, for the other term we do a similar expansion of the density.

$$\begin{aligned} & UQPE_{X_1}(\tau) \int_{\mathcal{Y}} K(u) f_Y(Q_Y[\tau] + hu) du \\ &= UQPE_{X_1} + \int_{\mathcal{Y}} K(u) \sum_{j=1}^{r-1} \frac{h^j u^j f_Y^{(j)}(Q_Y[\tau])}{j!} du + \frac{h^r}{r!} \int K(u) u^r f_Y^{(r)}(\tilde{Q}_Y[\tau]) du \\ &= UQPE_{X_1}(\tau) + O(h^r). \end{aligned}$$

Therefore, we obtain that the bias is of order $O(h^r)$:

$$E \left[\frac{1}{n} \sum_{i=1}^n K_h(Y_i - Q_Y[\tau]) \cdot [E[\beta_1(\xi_\tau(X))|Y = Y_i] - UQPE_{X_1}(\tau)] \right] = O(h^r).$$

Now, for the variance we have

$$\begin{aligned} & \text{Var} \left[\frac{1}{n} \sum_{i=1}^n K_h(Y_i - Q_Y[\tau]) \cdot [E[\beta_1(\xi_\tau(X))|Y = Y_i] - UQPE_{X_1}(\tau)] \right] \\ &= \frac{1}{nh^2} \text{Var} \left[K \left(\frac{Y_i - Q_Y[\tau]}{h} \right) [E[\beta_1(\xi_\tau(X))|Y = Y_i] - UQPE_{X_1}(\tau)] \right] \\ &\leq \frac{1}{nh^2} E \left[K \left(\frac{Y_i - Q_Y[\tau]}{h} \right)^2 [E[\beta_1(\xi_\tau(X))|Y = Y_i] - UQPE_{X_1}(\tau)]^2 \right] \\ &= \frac{1}{nh^2} \int K \left(\frac{y - Q_Y[\tau]}{h} \right)^2 [E[\beta_1(\xi_\tau(X))|Y = y] - UQPE_{X_1}(\tau)]^2 f_Y(y) dy \\ &= \frac{1}{nh} \int K(u)^2 [E[\beta_1(\xi_\tau(X))|Y = Q_Y[\tau] + hu] - UQPE_{X_1}(\tau)]^2 f_Y(Q_Y[\tau] + hu) du \\ &= \frac{1}{nh} \int K(u)^2 [hu E^{(1)}[\beta_1(\xi_\tau(X))|Y = \tilde{Q}_Y[\tau]]]^2 [f_Y(Q_Y[\tau]) + hu f_Y^{(1)}(\tilde{Q}_Y[\tau])] du. \end{aligned}$$

This implies that

$$\text{Var} \left[\frac{1}{n} \sum_{i=1}^n K_h(Y_i - Q_Y[\tau]) \cdot [E[\beta_1(\xi_\tau(X))|Y = Y_i] - UQPE_{X_1}(\tau)] \right] = O(n^{-1}h).$$

Therefore, we obtain that

$$\frac{1}{n} \sum_{i=1}^n K_h(Y_i - Q_Y[\tau]) \cdot [E[\beta_1(\xi_\tau(X))|Y = Y_i] - UQPE_{X_1}(\tau)] = O_p(h^r + n^{-1/2}h^{1/2}).$$

Under Assumption 4, this term is $o_p(n^{-1/2}h^{-1/2})$, since

$$(nh)^{1/2}O_p(h^r + n^{-1/2}h^{1/2}) = O_p((nh)^{1/2}h^r + h^{1/4}) = o_p(1),$$

since $(nh)^{1/2}h^r \rightarrow 0$, and $h \rightarrow 0$ as $n \rightarrow \infty$. Therefore, the bias term is

$$\sqrt{nh} \left(\frac{\frac{1}{n} \sum_{i=1}^n K_h(y_i - Q_Y[\tau]) \cdot [E[\beta_1(\xi_\tau(X))|Y = y_i] - UQPE_{X_1}(\tau)]}{\frac{1}{n} \sum_{i=1}^n K_h(y_i - Q_Y[\tau])} \right) = \frac{o_p(1)}{f_Y(Q_Y[\tau]) + o_p(1)} = o_p(1).$$

Combining this fact with (A.11), we obtain

$$\sqrt{nh} \left(\widehat{UQPE}_{X_1}(\tau) - UQPE_{X_1}(\tau) \right) \xrightarrow{d} N \left(0, \sigma_\tau^2(Q_Y[\tau]) f_Y(Q_Y[\tau])^{-1} \int K(u)^2 du \right).$$

In view of the result in Theorem 2, we have

$$\sqrt{nh} \left(\widehat{UQPE}_{X_1}(\tau) - UQPE_{X_1}(\tau) \right) \xrightarrow{d} N \left(0, \sigma_\tau^2(Q_Y[\tau]) f_Y(Q_Y[\tau])^{-1} \int K(u)^2 du \right).$$

A.5 Inference for the matching function

By Theorem 1, we have that

$$\hat{\xi}_\tau(x) - \xi_\tau(x) = -\frac{1}{x' \dot{\beta}(\xi_\tau(x))} \frac{1}{n} \sum_{i=1}^n x' \Psi_i(\xi_\tau(x)) + \frac{1}{x' \dot{\beta}(\xi_\tau(x))} \frac{1}{n} \sum_{i=1}^n \psi_i(\tau) + o_p(n^{-1/2}).$$

We use this influence function representation to estimate the asymptotic variance. First, consider the term $x' \dot{\beta}(\xi_\tau(x))$. Assumptions 1.1 and 1.2, allow us to write $F_{Y|X}(x' \beta(\eta)|x) = \eta$, so that $x' \dot{\beta}(\eta) = f_{Y|X}(x' \beta(\eta)|x)^{-1}$, and hence

$$x' \dot{\beta}(\xi_\tau(x)) = \frac{1}{f_{Y|X}(x' \beta(\xi_\tau(x))|x)}.$$

The function $\eta \mapsto f_{Y|X}(x' \beta(\eta)|x)^{-1}$ is usually called the sparcity function. It plays a central role in inference for quantile regression, see Section 3.4.2 in [Koenker \(2005\)](#), but also in (conditional)

mode estimation as in [Ota, Kato, and Hara \(2019\)](#). The estimator is

$$x' \widehat{\beta}(\widehat{\xi}_\tau(x)) = \left[\frac{x' \hat{\beta}(\hat{\xi}_\tau(x) + h_{KM}) - x' \hat{\beta}(\hat{\xi}_\tau(x) - h_{KM})}{2h_{KM}} \right]^{-1}.$$

In the case where either $\hat{\xi}_\tau(x) + h_{KM} > 1$ or $\hat{\xi}_\tau(x) - h_{KM} < 0$, the implementation is

$$x' \widehat{\beta}(\widehat{\xi}_\tau(x)) = \left[\frac{x' \hat{\beta}(\hat{\xi}_\tau(x) + \min\{h_{KM}, \tau_{\max} - \hat{\xi}_\tau(x)\}) - x' \hat{\beta}(\hat{\xi}_\tau(x) - \min\{h_{KM}, \hat{\xi}_\tau(x) - \tau_{\min}\})}{\min\{h_{KM}, \tau_{\max} - \hat{\xi}_\tau(x)\} + \min\{h_{KM}, \hat{\xi}_\tau(x) - \tau_{\min}\}} \right]^{-1},$$

where h_{KM} is the bandwidth of [Koenker and Machado \(1999\)](#), $\tau_{\min} = \epsilon$, and $\tau_{\max} = 1 - \epsilon$ for some small $1/2 > \epsilon > 0$.

Now we focus on the term $x' \Psi_i(\xi_\tau(x))$, where

$$\Psi_i(\eta) = E [f_{Y|X}(X'\beta(\eta)|X)XX']^{-1} (\eta - \mathbb{1}\{y_i \leq x'_i\beta(\eta)\}) x_i.$$

By [Powell \(1991\)](#), it can be estimated by

$$x' \widehat{\Psi}_i(\widehat{\xi}_\tau(x)) = x' \left[\frac{1}{2nh_p} \sum_{j=1}^n \mathbb{1}\{|y_j - x'_j \hat{\beta}(\hat{\xi}_\tau(x))| \leq h_p\} x_j x'_j \right]^{-1} (\hat{\xi}_\tau(x) - \mathbb{1}\{y_i \leq x'_i \hat{\beta}(\hat{\xi}_\tau(x))\}) x_i.$$

where h_p is given in Section 3.4.2 in [Koenker \(2005\)](#). Finally,

$$\psi_i(\tau) = f_Y(Q_Y[\tau])^{-1} (\tau - \mathbb{1}\{y_i \leq Q_Y[\tau]\})$$

can be estimated by

$$\widehat{\psi}_i(\tau) = \hat{f}_Y(\hat{Q}_Y[\tau])^{-1} (\tau - \mathbb{1}\{y_i \leq \hat{Q}_Y[\tau]\}).$$

So, the estimator of the asymptotic variance is

$$\frac{1}{n} \frac{1}{x' \widehat{\beta}(\widehat{\xi}_\tau(x))} \sum_{i=1}^n \left(-x' \widehat{\Psi}_i(\widehat{\xi}_\tau(x)) + \widehat{\psi}_i(\tau) \right)^2.$$

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