Dynamic analysis of linear Microeconomic expectations

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Abstract

Competitive markets tend to reach equilibrium through adjustments in demand and supply price and quantities.

Under uncertainty in market conditions, agents typically hold (or are forced to assume) certain expectations of future prices, based on past information, that contribute to justify production choices.

Expectation modeling alternatives include a set of choices. In this paper, we revisit the formal structure of extrapolative expectations. Under standard dynamic systems theory, we explore the solution and possible trajectories. We contribute to the complete analysis of certain critical case in an extension to this work.

Further analysis will tackle the translation of this result to other trajectories.

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1 Introduction

Competitive markets tend to reach equilibrium through adjustments in demand and supply price and quantities.

In many markets, there is intrinsic uncertainty on future prices or other conditions, which could affect outcomes. For instance, in many productive sectors (as in agriculture) supply decisions have to be made in advance under ignorance of the precise future production price. Agents on these market typically hold certain expectations of future price, based on past information, that contribute to justify production choices.

These expectations are usually modelled with some formulation. Both myopic and adaptive expectations are basic formulas for expectations ubiquitous in the literature. It is possible to develop more sophisticated behaviour like the assumed in extrapolative expectations, where agents facing uncertainty use past values from two periods to assume an expected price value.

1.1 A simple model

We define a simple partial competitive equilibrium model of supply and demand as follows: first, we consider a pair of affine functions

$$Q^D(P) = a + bP \tag{1}$$

and

$$Q^S(P) = A + BP \tag{2}$$

...where $Q^D(\bullet)$ and $Q^S(\bullet)$ are the *demand* and *supply* functions, respectively. The so-called TYPICAL CASE involves values for the coefficients fulfilling the conditions

$$a > 0 \tag{3}$$
$$b < 0 \tag{4}$$
$$B > 0 \tag{5}$$

and

$$A < a \tag{6}$$

On top of these function definitions, we have an EQUILIBRIUM CONDITION, given by

$$Q^D(P) = Q^S(P) \tag{7}$$

We define a market equilibrium as the pair (Q^*, P^*) such that

$$Q^* = Q^D(P^*) = Q^S(P^*) \tag{8}$$

Simple algebra shows that for our simple model

$$P^* = \frac{a-A}{B-b} = \frac{A-a}{b-B} \tag{9}$$

and

$$Q^* = \frac{aB-Ab}{B-b} \tag{10}$$

Notice that in a *typical case*, we will always have $P^* = \dfrac{A-a}{b-B} = \dfrac{a-A}{B-b} > 0$, while $Q^* = \dfrac{aB-Ab}{B-b} > 0$ only if additionally $aB > Ab$.

1.2 Price expectations

In many circumstances, microeconomic agents have to make decisions based on partial information of possible outcomes.

For instance, it can be argued that many suppliers make their production decisions without knowing for sure what will be the price for which they can sell their products.

This uncertainty can be dynamically modelled, assuming agents use past information to make an expectation of future price. Though this dynamic modelling could in principle be performed under a continuous time approach, the discrete time modelling is a well-suited approach to tackle such expectation formation.

We will thus continue our work under the formulation of a discrete process in which variables take values at times $t = 0, 1, \ldots$.

We could then reformulate our simple model under a dynamic framework, in the following way: there are time-dependant affine demand and supply functions given by

$$Q_t^D\left(P_t\right) = a + bP_t \tag{11}$$

and

$$Q_t^S\left(P_t^E\right) = A + BP_t^E \tag{12}$$

... where P_t is the prevailing price at moment t, and P_t^E is the price producers expect at time t, using possibly past information and an expectation formula.

Normally, this model is understood to behave in two different phases, having a short-run equilibrium at each moment t, and possibly converging (or not) to a long-run equilibrium in which $P_t \to P^*$. The market mechanism is as follows:

- Between time $t-1$ and t suppliers determine expected price P_t^E at time t, using available information.
- At time t, supply is given by the aggregation of individual producers as $\bar{Q} = Q_t^S\left(P_t^E\right)$, the instantaneous inelastic supply conditioned by compromised actions of producers before time t. A short-run equilibrium happens at a price P_t such that the market is clear, that is,

$$P_t \tag{13}$$

  such that

$$\bar{Q} = Q_t^D\left(P_t\right) \tag{14}$$

  For the typical case, this price P_t always exists and is unique, given by

$$P_t = \frac{\bar{Q} - a}{b} \tag{15}$$

- As time evolves, a trajectory P_t is formed. Eventually, as $t \to \infty$, a long-run state is build; this trajectory could be *asymptotically stable*, if $P_t \to \bar{P}$ for some long-run equilibrium price $\overline{P}$, or not, otherwise.

If the price expectation formula is known, the model can in principle be solved by looking for a solution that gives us the price trajectory at each moment t. This solution generally depends on some base (*initial*) conditions [normally given as price values at initial moments $P_0, P_1, \ldots$]. Since the analysis centers around an equation involving prices in different discrete moments of time, normally a recursive equation[1] is formed.

1.3 Basic expectation formulas

The literature has many expectation formulations used to model this types of interaction. Two very well-known models that have been used to model price expectations are that of myopic expectations [3, 9]:

$$P_t^E = P_{t-1} \tag{16}$$

[1] Also called difference equation, since most times the equation can be rewritten using the discrete difference operators $\Delta P_t, \Delta^2 P_t$, etc.

and the adaptive expectations [6, 7]:

$$P_t^E = \delta P_{t-1} + (1-\delta)\, P_{t-1}^E \tag{17}$$

$$= P_{t-1}^E + \delta \left(P_{t-1} - P_{t-1}^E\right) \tag{18}$$

Both of them can be solved (under a linear market model) by the resolution of a linear first-order recursive equation.

1.4 Extrapolative expectations

Both previous expectation models use only information from the immediately preceding point of time. We develop here a logical extension to this, which is an expectation formula that uses information from at least two previous periods. The most straightforward extension is given by the EXTRAPOLATIVE EXPECTATIONS [5, 2], which for our purposes can be written as

$$P_t^E = \eta P_{t-1} + (1-\eta)\, P_{t-2} \tag{19}$$

... so that it derives the expected price as a weighted average of the last two observed market prices.

Extrapolative expectations are not only a reasonable use of using more information than the simplest models. As Negeshi [5] argues, they are a reasonable balance (for example, vis-à-vis the idea of rational expectations [4]) that uses costly information in an efficient way to form price expectations that have interesting theoretical properties, such as providing a dynamic framework for stable price adjustements.

Introducing (19) into (14), we get

$$P_t = \frac{B\eta P_{t-1}}{b} + \frac{B\,(1-\eta)\, P_{t-2}}{b} + \frac{A-a}{b} \tag{20}$$

... which is a linear, second-order recursive equation with constant coefficients, a type of recursive equation which in more general form can be expressed

$$P_t = \alpha P_{t-1} + \beta P_{t-2} + \gamma \tag{21}$$

Again, if $\gamma = 0$ in (21), we would have an HOMOGENEOUS linear recursive equation; otherwise, the equation is NON-HOMOGENEOUS.

To elucidate the complete behaviour of this model, we will use standard dynamic analysis

of linear recursive equations.

2 Resolution of equilibrium under extrapolative expectations

2.1 Some intermediate results

It is well-known ([1, 8]) that a second-order linear

$$y_t = \alpha y_{t-1} + \beta y_{t-2} + \gamma \tag{22}$$

can be solved by finding the complete set of solutions for the homogeneous case of (22)

$$y_t = \alpha y_{t-1} + \beta y_{t-2} \tag{23}$$

and adding to this a particular solution y_t^p that is known to solve (22).

2.2 Solutions to the homogeneous equation

We proceed to develop the complete family of solutions for the homogeneous case in (23). Some knowledge in recursive equations hints at trying a solution of the form

$$y_t = Cr^t \tag{24}$$

with which (23) turns into

$$Cr^t - \alpha Cr^{t-1} - \beta Cr^{t-2} = 0 \tag{25}$$

Assuming $r \neq 0$, we divide both sides of (25) by Cr^{t-2}; we can then rewrite the equation as

$$r^2 - \alpha r - \beta = 0 \tag{26}$$

... that will always have two (not necessarily different or real) roots r_1 and r_2.

Equation (26) is the so-called CHARACTERISTIC EQUATION of the original recursive equation, and shows three different cases, considering the sign of the DISCRIMINANT $\Delta = \alpha^2 + 4\beta$:

I) $\Delta > 0$, and r_1 and r_2 are real distint roots of (26).

II) $\Delta = 0$, and $r_1 = r_2$ is the real repeated root of (26).

III) $\Delta < 0$, and r_1 and r_2 are the complex conjugate roots of (26), with $r_{1,2} = a \pm bi$.

In the first case, we note that both Cr_1^t and Cr_2^t are linearly independent, and, since both expressions solve (23), the complete homogeneous family of solutions is given by

$$y_t^h = C_1 r_1^t + C_2 r_2^t \tag{27}$$

... that has two undetermined coefficients C_1, C_2 to be determined with initial conditions[2].

In the second case, with $r_1 = r_2 = r$, it becomes obvious that $C_1 r_1^t$ and $C_2 r_2^t$ are not linearly independent. We must find a family of solutions linearly independent of $C_1 r_1^t$ to build the complete set of solutions to 23. We notice that

$$y_t = Ctr^t \tag{28}$$

also solves (23) when $\Delta = 0$, and is linearly independent of Cr^t; hence, we conclude that the complete set of solutions for (23) is given in this case by

$$y_t^h = C_1 r^t + C_2 t r^t \tag{29}$$

The third case involves two complex conjugate roots

$$r_1 = a + bi \tag{30}$$

and

$$r_2 = a - bi \tag{31}$$

We can propose a solution of the form

$$y_t^h = C_1 r^t + C_2 t r^t \tag{32}$$

$$= C_1 (a + bi)^t + C_2 (a - bi)^t \tag{33}$$

[2]It is a well-known fact that every homogenous linear recursive equation [or order n] of these types will have n linearly independent solutions; in other words, the family of solutions for the homogeneous equation, a vector functional space, has an n-dimensional basis. We won't show this result here. The reader could consult this technical result in [1]

Transforming the complex roots to their polar form, we get

$$a + bi = \rho\left(\cos\theta + i\sin\theta\right) \tag{34}$$

and

$$a - bi = \rho\left(\cos\theta - i\sin\theta\right) \tag{35}$$

and using Euler's formula for complex numbers:

$$e^{\theta i} = \cos\theta + i\sin\theta \tag{36}$$

we could get

$$\left(\cos\theta + i\sin\theta\right)^t = \cos\left(t\theta\right) + i\sin\left(t\theta\right) \tag{37}$$

Replacing in (33), we get

$$y_t^h = C_1\rho^t\left(\cos\left(\theta t\right) + i\sin\left(\theta t\right)\right) + C_2\rho^t\left(\cos\left(\theta t\right) - i\sin\left(\theta t\right)\right) \tag{38}$$

$$= C_1\rho^t e^{\theta t i} + C_2\rho^t e^{-\theta t i} \tag{39}$$

To summarise all these deductions, we could briefly define the homogeneous solution as

$$y_t^h = \begin{cases} C_1 r_1^t + C_2 r_2^t & \text{if } \alpha^2 + 4\beta > 0 \\ C_1 r_1^t + C_2 t r_1^t & \text{if } \alpha^2 + 4\beta = 0 \\ C_1\left(a + bi\right)^t + C_2\left(a - bi\right)^t & \text{if } \alpha^2 + 4\beta < 0 \end{cases} \tag{40}$$

... where $r_1 = \frac{\alpha}{2} + \frac{\sqrt{\alpha^2+4\beta}}{2}$, $r_2 = \frac{\alpha}{2} - \frac{\sqrt{\alpha^2+4\beta}}{2}$, $a = \frac{\alpha}{2}$ and $b = \frac{\sqrt{-\alpha^2-4\beta}}{2}$

2.3 Particular solution

To find the complete set of solutions to (22), we can add a particular solution to (22) to the previous homogeneous solutions.

The simplest possible solution to (22) is given by a constant solution. If we assume $y_t = \bar{y}$, we will have

$$\bar{y} - \alpha\bar{y} - \beta\bar{y} = \gamma \tag{41}$$

from which we get

$$\bar{y} = \frac{\gamma}{1-\alpha-\beta} \tag{42}$$

This will work as the particular solution as long as $1-\alpha-\beta \neq 0$, in which case $\bar{y}$ would not even be defined in (42).

In the case that $\alpha+\beta=1$, we could try the next-simpler solution, the linear function $y_t = t\bar{y}$. Replacing this solution in (22), we obtain

$$\bar{y}\left(t-\alpha t+\alpha-\beta t+2\beta\right) = \gamma \tag{43}$$

Taking into account $\alpha+\beta=1$, we get

$$\bar{y}\left(\alpha+2\beta\right) = \gamma \tag{44}$$

$$\bar{y}\left(1+\beta\right) = \gamma \tag{45}$$

$$\bar{y} = \frac{\gamma}{1+\beta} \tag{46}$$

This of course works, of course again, as long as $\beta \neq -1$. If we had $\beta=-1$ while at the same time $\alpha+\beta=1$, then clearly in this case

$$\alpha = 2 \tag{47}$$

and

$$\beta = -1 \tag{48}$$

We try in this case the next-simpler solution, quadratic $y_t = \bar{y}t^2$. We introduce it in (22), considering (47) and (48), to obtain:

$$\bar{y}t^2 - 2\bar{y}\left(t-1\right)^2 + \bar{y}\left(t-2\right)^2 = \beta \tag{49}$$

$$\bar{y}\left(t^2-2t^2+4t-2+t^2-4t+4\right) = \beta \tag{50}$$

$$2\bar{y} = \beta \tag{51}$$

from which

$$y_t^p = \frac{\beta}{2}t^2 \tag{52}$$

To sum up, the particular solution is given by

$$y_t^p = \begin{cases} \dfrac{\gamma}{1-\alpha-\beta} & \text{if } \alpha+\beta \neq 1 \\ \dfrac{\gamma t}{1+\beta} & \text{if } \alpha+\beta = 1 \text{ and } \beta \neq -1 \\ \dfrac{\beta}{2}t^2 & \text{if } \alpha = 2 \text{ and } \beta = -1 \end{cases} \tag{53}$$

We call $\dfrac{\gamma}{1-\alpha-\beta}$, $\dfrac{\gamma t}{1+\beta}$ and $\dfrac{\beta}{2}t^2$ the *stationary*, *linear* and *quadratic* particular solutions, respectively.

2.4 General solution

To solve for the general solution to (22), we add the corresponding homogeneous and particular solutions.

Let us consider the following

Example 2.1. Let us consider a competitive market under extrapolative expectations, following the recursive equation as in (20). Let $B = 0.8$, $b = -0.7$, $A = 0$, $a = 120$, $\eta = 0.87$. Find the price trajectory knowing that $P_0 = 40$ and $P_1 = 80$.

We notice here that $\Delta = \alpha^2 + 4\beta = \left(\frac{B\eta}{b}\right)^2 + 4\frac{B(1-\eta)}{b} > 0$, and $\alpha+\beta = \frac{B\eta}{b} + \frac{B(1-\eta)}{b} \neq 1$. Hence, the general solution will be given by the sum of

$$P_t^h = C_1 r_1^t + C_2 r_2^t = C_1 \left(-\frac{87}{175} + \frac{\sqrt{3019}}{175}\right)^t + C_2 \left(-\frac{87}{175} - \frac{\sqrt{3019}}{175}\right)^t \tag{54}$$

and

$$P_t^p = \frac{\gamma}{1-\alpha-\beta} = \frac{A-a}{b-B} = 80 \tag{55}$$

The two constants C_1 and C_2 can be retrieved so as to make the trayectory compatible with the initial conditions, that is

$$C_1 + C_2 = P_0 \tag{56}$$

and

$$C_1 r_1 + C_2 r_2 = P_1 \tag{57}$$

from which

$$C_1 = 20 + \frac{8740\sqrt{3019}}{3019} \tag{58}$$

and

$$C_2 = 20 - \frac{8740\sqrt{3019}}{3019} \tag{59}$$

Example 2.2. Let us consider a second set of parameters for a competitive market under extrapolative expectations, following the recursive equation as in (20). Let $B = 1$, $b = -0.8$, $A = -6$, $a = 120$, $\eta = 0.2$. Find the price trajectory knowing that $P_0 = 40$ and $P_1 = 80$.

Here clearly $\Delta = \alpha^2 + 4\beta = \left(\frac{B\eta}{b}\right)^2 + 4\frac{B(1-\eta)}{b} < 0$; clearly also $\alpha + \beta = \frac{B\eta}{b} + \frac{B(1-\eta)}{b} \neq 1$. Hence, the homogeneous solution has two complex conjugates as roots in the characteristic equation. The general solution will be given by the sum of

$$P_t^h = C_1 r_1^t + C_2 r_2^t = C_1 \left(-\frac{1}{8} + \frac{3\sqrt{7}i}{8}\right)^t + C_2 \left(-\frac{1}{8} - \frac{3\sqrt{7}i}{8}\right)^t \tag{60}$$

and

$$P_t^p = \frac{\gamma}{1-\alpha-\beta} = \frac{A-a}{b-B} = 70 \tag{61}$$

The two constants C_1 and C_2 can again be computed so as to make the trayectory compatible with the initial conditions, so that

$$C_1 + C_2 = P_0 \tag{62}$$

and

$$C_1 r_1 + C_2 r_2 = P_1 \tag{63}$$

It is quite straightforward that $C_1 = \overline{C_2}$ (constants must be complex conjugate). We have

$$C_1 = 20 - \frac{340\sqrt{7}i}{21} \tag{64}$$

and

$$C_2 = 20 + \frac{340\sqrt{7}i}{21} \tag{65}$$

2.5 Trajectories

Finding a solution for the recursive equation equates solving the equation and finding the corresponding sum $y_t^h + y_t^p$. To find the complete trajectory, initial conditions y_0, y_1 must be provided. A solution can be then viewed as a function

$$y_t = f(t, y_0, y_1, \alpha, \beta, \gamma) \tag{66}$$

If the parameters α, β, γ are implied to have fixed values, we can simply write the solution as

$$y_t = f(t, y_0, y_1) \tag{67}$$

If $f(t, y_0, y_1) = Y_t$, and $f(t+1, y_0, y_1) = Y_{t+1}$, we can define the successor function

$$F(y_t, 1) = f(t+1, y_0, y_1) \tag{68}$$

and its extension "element k times ahead"

$$F(y_t, k) = f(t+k, y_0, y_1) \tag{69}$$

Notice that the successor function can be replaced by $\alpha y_t + \beta y_{t-1} + \gamma$, if we also know y_{t-1}, of course.

2.6 Stability considerations

A so-called steady state is a situation in which certain stable stage has been reached, or if the trajectory has not reached the stage yet, it approaches it.

In differential equations' theory, an EQUILIBRIUM is a point $\hat{y} = y(t)$ such that all involved derivatives of y w.r.t. time are null evaluated at $\hat{y}$, so the system stays stable in this equilibrium.

The equivalent concept in recursive equations is the following:

Definition 2.3. Let $y_t = \alpha y_{t-1} + \beta y_{t-2} + \gamma$ be certain recursive equation. $\hat{y}$ is an EQUILIBRIUM of the equation if there is a number of periods m such that a trajectory having for certain moment T $y_{T+1} = y_{T+2} = \ldots y_{T+m} = \hat{y}$, will have $y_P = \hat{y}$ for any $P > T$.

Lemma 2.1. *The equation* (20) *has the long-run price equilibrium* $P^* = \dfrac{A-a}{b-B}$ *as an equilibrium.*

Proof. Setting $P_0 = P_1 = \dfrac{A-a}{b-B}$ in equation (20), we get $P_2 = P_3 = \ldots = \dfrac{A-a}{b-B}$. □

Lemma 2.2. *A second-order recursive equation as in* (22) *can have at most one equilibrium* $\hat{y}$.

We leave the proof of lemma 2.2 to the reader. As a hint, notice that by construction, $\lim_{t\to\infty} y_t^h$ is either zero or one infinite form. Consider then that if the equation has an equilibrium, then the particular solution y_t^p must be of the form $\frac{\gamma}{1-\alpha-\beta}$, which is unique.

The fact that every market with extrapolative expectations has the long-run price as equilibrium under the typical case does not imply, of course, that every possible trajectory will converge to this equilibrium.

Definition 2.4. An equilibrium point $\hat{y}$ for (22) is said to be STABLE if, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|y - \hat{y}| < \epsilon \implies |F(y,k) - \hat{y}| < \delta \tag{70}$$

In plain language, an equilibrium is stable if trajectories relatively near the equilibrium stay near the equilibrium.

Definition 2.5. We say that an equilibrium point $\hat{y}$ is UNSTABLE if it is not stable.

Another useful stability concept is that of an asymptotically stable point. Firstly, we define the following:

Definition 2.6. A trajectory $y_t = f(t, y_0, y_1)$ is said to CONVERGE TO $\hat{y}$ if

$$\lim_{t\to\infty} y_t = \hat{y} \tag{71}$$

that is,

$$\forall \epsilon > 0 \quad \exists n : \forall m \geq n \ |y_m - \hat{y}| < \epsilon \tag{72}$$

Definition 2.7. A point $\hat{y}$ is ATTRACTING if there exists $\epsilon > 0$ for which every trajectory that intersects the ball

$$\mathcal{B}(\hat{y}, \epsilon) = \{x \in \mathbb{R} : \ |x - \hat{y}| < \epsilon\} \tag{73}$$

converges to $\hat{y}$. If this is true for *every* $\epsilon > 0$, we say that $\hat{y}$ is GLOBALLY ATTRACTING.

Sometimes, a particular set of attracting points is called an ATTRACTOR.

Now, we address the aforementioned concept:

Definition 2.8. An equilibrium point $\hat{y}$ is ASYMPTOTICALLY STABLE if it is stable and attracting. If $\hat{y}$ is stable and globally attracting, we say that it is GLOBALLY ASYMPTOTICALLY STABLE.

The stability character of a trajectory and a reference equilibrium depends on the characteristics of the solved equation, particularly the properties of the particular and homogeneous solutions.

2.7 Relevant parameter space

It is easy to see that these characteristics depend on the values of the parameters α and β in (22) [or in (21)].

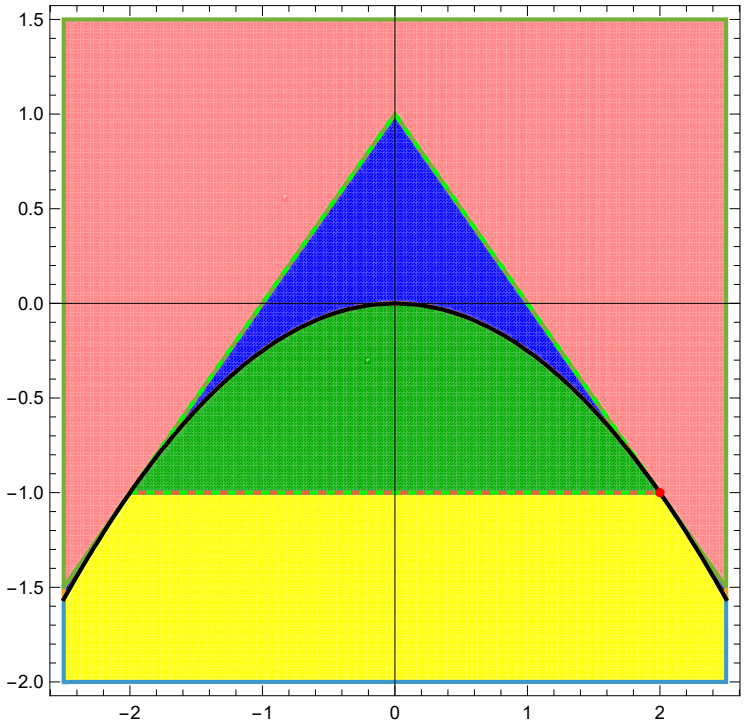

Figure 1: Relevant parameter space α-β. Characterisation of trajectories.

Hence, we could analyse the main characteristics of every possible trajectory with the graph in figure 1.

In that figure, the parabola $\alpha^2 + 4\beta$ defines the precise separation between the lower lob of the parabola (in which discriminant Δ of the characteristic equation is negative, and there are a pair of complex conjugate roots) and the upper part above the parabola, in which there are two distinct real roots for the characteristic equation. On the parabola, we

have the case of a null discriminant, and so there the characteristic equation has a repeated real root.

There is a line with slope -1 touching the parabola in only one point: $(2, -1)$. Over this line (except in the mentioned point) there is a linear particular solution. On the point, however, we have the case of the quadratic particular solution.

Outside this line, the particular solution is the stationary $y_t^p = \frac{\gamma}{1-\alpha-\beta}$.

There is a blue triangular zone over the parabola; this region represents asymptotic stability of the long-run equilibrium. This is also the case of the green region inside the hypograph of the parabola.

The pink zone above the triangular brown region corresponds to (asymptotically) unstable equilibria under the case of two different real roots. Similarly, the yellow zone below the pink area involves unstable equilibria under the case of complex conjugate roots.

What remains to consider are points over the two straight segments from $(-2, -1)$ to $(0, 1)$ and from $(0, 1)$ to $(2, -1)$, on one hand, and the limit horizontal segment below the green cap from $(-2, -1)$ to $(2, -1)$.

The segment from $(0, 1)$ to $(2, -1)$ includes the extreme point $(2, -1)$. This point is the only instance of a quadratic particular solution, under which the long-run equilibrium is unstable.

The left-most segment from $(-2, -1)$ to $(0, 1)$, as well as the horizontal segment inside the parabola with $\beta = 1$ are very special cases: these characterise trajectories with non-explosive behaviours [the long-run equilibrium is not asymptotically unstable] but neither is the equilibrium attracting. Included in these peculiar instances, which we call CRITICAL CASES for lack of a more precise term, are periodic orbits, as well as a-periodic trajectories as the one of example 2.2.

3 Further analysis

3.1 Some questions

Having used well-known elements of linear dynamic theory to solve the initial problem and obtain the analytical expected trajectories, we are now interested in posing some motivational questions that drive further analysis:

- Which will be trajectory of price over time? Which parameters affect it and how?
- How different trajectories show in the Lémeray diagram?
- Which will be the image of $f(t, y_0, y_1, \alpha, \beta, \gamma)$?
- If it were known that certain $y_t \in \mathrm{Im}(f)$, how can we work out the time at which the trajectory passes through y_t? [This is the problem of inverting the trajectory.]

- If we know two values y_m, y_n are in a trajectory, how can we find out the difference in time $|m - n|$?
- If certain trajectory passes through a point y_t in its image at moment t, under which circumstances will it not fall back to y? (We call the property of not getting back to a previous image point the *no-fall-back* property. The literature also refers to this concept with the definition of a non-recurring trajectory.

Some of these questions are known to be hard to answer in a general way. Some have easier answers. We can develop the answers to all these answers for a complex particular case, and try to extrapolate the most we can to the general complex conjugate case, and later to the realm of different real roots. We propose this as an extension to this work.

The no-fall-back property is easy to study when the roots of the characteristic equation are real; in the case of a negative discriminant Δ the analysis is certainly more demanding.

3.2 Critical cases

In figure 1 we described the α, β parameter space and the division of it in different areas with diverse dynamic properties.

These borderline cases arise when the properties of the trajectories are such that values are restricted to a certain set or interval, but do not asymptotically converge to any equilibrium. In particular, we are interested in such cases when the particular solution is stationary, and the roots of the characteristic equation are distinct real numbers of complex conjugates. We so define the following:

Definition 3.1. An instance of (22) is called a CRITICAL CASE if

a) $\Delta > 0$ (real distinct roots), $\alpha < 0$ and $\beta = \alpha + 1$ (the max of absolute values of roots is exactly 1 under the stationary particular solution), or

b) $\Delta < 0$ (complex conjugate roots) and $\beta = -1$ (the modulus of both roots is exactly 1).

3.3 Future extensions

Though we do not explore it here, it can be shown that the critical complex case can be easily solved by means of complex algebra. We propose to extend these findings to generic complex cases.

We will expand this idea in future research, as well as the applications for the case of real, distinct roots.

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