# On Collusion and the Number of Firms*

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**Abstract**

Collusion remains a recurrent feature of many markets, fueled by the promise of supra-competitive profits. A large body of theory predicts that cartels become less likely as the number of firms rises. We show, by contrast, that under broad conditions the incentives to collude can *increase* with market size. The mechanism is simple: competitive profits fall faster than collusive profits as additional firms enter, raising the relative reward from coordination. This mechanism challenges a core premise of U.S. and European antitrust policy and, at the same time, helps reconcile theory with empirical evidence of cartels that comprise many participants. The result holds for both price and quantity competition, regardless of whether the self-enforcement constraint binds.

## 1 Introduction

Antitrust authorities routinely assert that the likelihood of collusion declines as the number of firms in a market increases (see, e.g., the European Commission and U.S. Department of Justice

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guidelines). In merger review, for example, the analysis of "coordinated effects" typically concludes that mergers are less likely to foster collusion when more firms remain active post-merger. This view is supported by a broad economic literature that identifies a range of factors that influence the stability of collusion. The prevailing conclusion in that literature is that, *ceteris paribus*, sustaining collusion becomes more difficult as the number of firms increases (Stigler, 1964; Selten, 1973; Kwoka & White, 1989; Motta, 2004; Harrington *et al.*, 1996; Ivaldi *et al.*, 2007; Fabra & Motta, 2018). Common explanations point to increased complexity in coordination, greater difficulty in monitoring and punishing deviations, and weaker incentives to maintain cooperation. While these arguments remain relevant, we challenge the standard view by highlighting a general—and often overlooked—mechanism through which the profitability, and therefore the appeal, of collusion may actually increase with the number of firms. Whether this countervailing effect dominates is an empirical question, which lies beyond the scope of this paper. Since the empirical record on the relationship between cartelization and the number of firms remains mixed, our framework offers a theoretical justification for studies that find a positive association between firm count and cartel formation.

We start with a general framework in which symmetric firms with constant marginal costs compete in the market. Within this setting, we focus on firms' incentives to form collusive agreements that generate supra-competitive profits. Identifying which agreements are sustainable requires checking that each firm has a unilateral incentive to comply—a condition formalized by the incentive compatibility, or *no-cheating*, constraint. This constraint is typically written as a threshold value for the discount factor: firms must be sufficiently patient for collusion to be self-enforcing. This "critical discount factor" approach is widely used in the literature to analyze how various factors affect the sustainability of collusion. In this paper, we revisit that approach and argue that it can offer an incomplete picture. Its conclusions often rely on strong assumptions about how the gains from collusion behave when the set of sustainable agreements shifts—assumptions that may lead to misleading inferences about how the number of firms influences the viability of collusion.[1]

The general framework provides a basis for computing the gains from collusion. To do so, we begin by characterizing the competitive benchmark.[2] We then define the profits attainable under a symmetric collusive agreement on the strategic variable, assuming firms follow a symmetric

[1]Our analysis focuses exclusively on the economic forces tied to the profitability of collusion and the incentive compatibility constraint. We do not dispute other well-established arguments that associate larger numbers of firms with challenges in monitoring, punishing deviations, or coordinating stable agreements.

[2]Throughout, "competitive" refers to the non-cooperative outcome under imperfect competition, such as Cournot or Bertrand equilibria.

trigger strategy. At this stage, we depart from the traditional analysis, which typically asks: *Given a desired collusive allocation, what is the minimum discount factor that renders it incentive compatible?* Instead, we reverse the question: *Given a fixed discount factor and number of firms, what is the most profitable symmetric allocation that satisfies the no-deviation constraint?* This shift lets us determine which collusive outcomes can be sustained, given firms' level of patience. Using this approach, we compute the gains from collusion—defined as the improvement in firm profits relative to the competitive baseline. These gains can be expressed in either absolute terms (as the difference in profits) or relative terms (as the ratio of the difference in profits to competitive profits).[3]

With this approach we show that, under mild assumptions, the *absolute gains* from collusion are non-monotonic in the number of firms, whereas the *relative gains* increase monotonically. The effect of the number of firms operates mainly through a single central mechanism, although the self-enforcing condition adds an additional constraint when firms are sufficiently impatient. The key mechanism is the gap between collusive and competitive profits: both decline as $n$ rises, but competitive profits typically fall faster. Consequently, the net gain from collusion can rise with $n$ —at least over some range— because the non-collusive benchmark becomes increasingly unattractive. A back-of-the-envelope Cournot calculation illustrates the point: a cartel in a market with two firms raises profits from \$16 to \$18, that is a gain of \$2 or an increase of 12.5%, meanwhile a cartel in a market with eleven firms raises profits from \$1 to \$3.27, that is a gain of \$2.27 or an increase of 227.27%.[4]

This result reflects basic economic forces and holds independently of the discount factor. However, when firms are impatient (that is, when the discount factor is low), collusion must be self-enforcing: the present value of future collusive profits must be large enough to deter unilateral deviation. In this case, the number of firms affects the incentive constraint in two distinct ways. First, as before, it raises the relative benefit of collusion by making the competitive benchmark less attractive. Second, it tightens the set of implementable collusive outcomes by reducing per-firm collusive profits, thereby weakening the threat of future punishment. This second effect can limit the profitability and enforceability of collusion as the number of firms increases, though it does not

[3]We abstract from the explicit costs of collusion. If such costs are fixed, then the absolute gain—as emphasized in Werden & Baumann (1986)—is the appropriate measure. If costs scale with collusive profits or reflect required investments in enforcement or coordination, the relative gain may be more informative. Accordingly, we report results using both metrics without taking a position on which is more appropriate.

[4]These figures assume zero production cost and market demand $Q = 12 - P$, Figure 1 shows that as the number of firms $n$ increases, the per-period competitive profit $144/(n+1)^2$ falls faster than the equal per-firm share of monopoly profit $36/n$.

necessarily eliminate the incentive to coordinate. In fact, under standard Cournot competition, a market with five firms creates stronger incentives to collude than a duopoly, for any discount factor. Within that framework, the highest absolute gains from collusion are often observed in markets with three or four firms. Taken together, these forces imply a non-monotonic relationship between the number of firms and the likelihood of collusion. By contrast, the relative gains from collusion increase monotonically with the number of firms for any discount factor.

Our findings contrast with several well-established results in the literature. For example, in the standard Bertrand model with homogeneous products, constant marginal costs, and perfect information, two key conclusions emerge: (i) the gains from collusion decline as the number of firms increases; and (ii) the critical discount factor needed to sustain collusion rises with the number of firms (i.e., $\underline{\delta} = 1 - 1/n$), narrowing the set of incentive-compatible agreements. However, these conclusions rely heavily on the model's simplifying assumptions. When we move beyond this framework, the relationship between the number of firms and collusion changes. In the standard Cournot model, for instance, the gains from collusion do not decline with the number of firms. Similarly, in models of price competition with private information about costs or with product differentiation, the predictions align more closely with our findings.[5] To illustrate this robustness, we present a model of price competition with differentiated products, where a parameter governs the degree of substitutability. As substitutability increases, the model converges to the standard Bertrand case and beyond a threshold some properties replicate. While absolute gains from collusion decline with both the number of firms and the level of substitutability, relative gains increase along both dimensions. This provides further support for the generality of our results—even in simplified environments.

A few studies have noted that collusion may become easier to sustain as the number of firms increases (Compte *et al.*, 2002; Asker & Nocke, 2021; Hatfield *et al.*, 2020, 2019). However, these results typically rely on additional assumptions—such as capacity constraints, syndicate formation, or the presence of social norms to support coordination. The only prior work we found that directly aligns with our result is Werden & Baumann (1986), who present an example in which collusion becomes easier to sustain as the number of firms grows. Our approach differs from theirs in two key ways. First, they consider only Cournot competition, whereas we offer a more general analysis that includes both quantity and price competition. Second, their model focuses on explicit collusion and does not incorporate the no-cheating constraint. In contrast, we show that their main insights are robust in a setting of implicit collusion, where the incentive compatibility

[5]While our analysis focuses on price and quantity competition, the logic extends to other strategic variables such as advertising, capacity, quality, and product durability (e.g., bulb life).

constraint must be incorporated.

Our contribution is twofold. First, we show that the standard critical discount factor approach can, in some cases, lead to misleading conclusions. For this reason, we adopt an alternative cost–benefit framework that focuses on identifying the most profitable implementable outcomes for a given level of patience. Second, we demonstrate that the incentives to collude may increase with the number of firms—a result that challenges the conventional wisdom in the literature. This finding underscores that the trade-off between the benefits and costs of collusion varies with market size in a non-trivial way. In practice, both sides of this trade-off may rise as the number of firms increases. As a result, the number of firms that maximizes the likelihood of cartel formation depends on the underlying market environment and does not admit a simple or universal characterization.

The remainder of the paper is organized as follows. Section 2 introduces a general framework that can be applied across a range of models. Sections 3, 4, and 5 develop specific applications: Cournot competition, Bertrand competition with differentiated products, and Bertrand competition with private information about marginal costs, respectively. Section 6 concludes.

## 2 General Framework

Consider an infinitely repeated industry with $n > 1$ symmetric firms, each with constant marginal costs.[6] The demand in each period for firm $i$ can be represented by $q_i(p_1, p_2, ..., p_n)$ or by the inverse demand function $p_i(q_1, q_2, ..., q_n)$, depending on the particular specification. In each period, each firm simultaneously chooses a strategic variable $\alpha \in \mathbb{R}$. While our analysis is motivated by models of price and quantity competition, the framework applies to other strategic variables as well. The examples below provide more structure for each model. When $n$ firms compete, we assume that there is a unique symmetric equilibrium $\alpha^*$. Competitive profits are $\pi^*(n)$ for each firm, and we assume that $n \cdot \pi^*(n)$ decreases with $n$, reflecting the idea that competition reduces industry profits.

Firms may collude by selecting a vector $(\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n$. Because we focus on symmetric equilibria, we let $\alpha \in \mathbb{R}$ denote the common choice of each firm, which yields per-period profit $\pi^c(\alpha, n)$ to every firm.[7] Profits vary with $\alpha$. Let $\alpha^+$ denote the most aggressive

[6] For analytical convenience, we allow $n$ to take any real value greater than one. The main results remain valid when $n$ is restricted to the set of natural numbers.

[7] We abstract from side payments or transfers among firms. In our framework, this assumption is trivial, but it

symmetric collusive target—the value that achieves the monopoly outcome and splits total profit evenly: $\pi^c(\alpha^+, n) = \Pi^{\mathrm{M}}/n$, where $\Pi^{\mathrm{M}} := \max_{\{\alpha_i\}_{i=1}^n} \sum_{i=1}^n \pi_i(\alpha_1, \ldots, \alpha_n)$. [8] We assume $\alpha^+$ is unique for every $n$. Define $\alpha^-$ as the symmetric value at which collusion and competition yield the same profit: $\pi^c(\alpha^-, n) = \pi^*(n)$. Both $\alpha^-(n)$ and $\alpha^+(n)$ depend on $n$. Hence, we define $\mathcal{A}_n := \left[\min\{\alpha^+, \alpha^-\}, \max\{\alpha^+, \alpha^-\}\right]$ as the interval of relevant collusive agreements given $n$, and $\mathcal{A} := \cup_{n=1}^{\infty} \mathcal{A}_n$. In Bertrand competition with differentiated products we have $\alpha^- \leq \alpha^+$, whereas in the Cournot model $\alpha^- \geq \alpha^+$. Throughout, we assume $\pi^c(\alpha, n)$ is monotone in $\alpha \in \mathcal{A}_n$. Given a collusive agreement $\alpha$, the number of firms $n$, and discount factor $\delta$, the present value of the per-firm gain from collusion relative to competition is:

$$\Big(\pi^c(\alpha, n) - \pi^*(n)\Big)\,(1 + \delta + \delta^2 + \delta^3 + \cdots). \tag{1}$$

The sustainability of any collusive agreement depends on firm's incentives to comply with the common target $\alpha$. If a firm unilaterally deviates, it earns, in that period, the profit $\pi^d(\alpha, n)$. By definition of equilibrium, $\pi^d(\alpha^-, n) = \pi^c(\alpha^-, n) = \pi^*(n)$; we also assume that $\pi^d(\alpha, n)$ is monotone in $\alpha \in \mathcal{A}_n$. For all $\alpha \in \mathcal{A}_n$ we have $\pi^d(\alpha, n) \geq \pi^c(\alpha, n) \geq \pi^*(n)$, with strict inequality if $\alpha \neq \alpha^-$. Throughout we assume that, for every $n$, both $\pi^c(\alpha, n)$ and $\pi^d(\alpha, n)$ are continuous in $\alpha \in \mathcal{A}_n$ and differentiable in $\alpha \in \mathrm{int}(\mathcal{A}_n)$. For simplicity, we consider only the standard trigger-strategy framework with perfect monitoring: once a deviation is detected, play reverts permanently to the competitive equilibrium. Given a common discount factor $\delta \in (0, 1)$ and a number of firms $n$, a collusive agreement $\alpha$ is honored if the following *no-cheating* incentive constraint holds:

$$\pi^c(\alpha, n)(1 + \delta + \delta^2 + \delta^3 + ...) \geq \pi^d(\alpha, n) + \pi^*(n)(\delta + \delta^2 + \delta^3 + ...) \tag{2}$$

We call a symmetric agreement $\alpha \in \mathcal{A}_n$ *implementable* if it satisfies (2). The benefits in (1) are achieved only when the agreement is implementable. The traditional approach defines the minimal discount factor $\underline{\delta}$ that allows an agreement $\alpha$ to be implementable.

**Proposition 1** *Given trigger-strategies with reverse to symmetric Nash equilibrium, then:*

*(i) The minimal discount factor that makes a collusive agreement $\alpha \in \mathcal{A}_n$ implementable in a market with $n$ firms is defined by $\underline{\delta}(\alpha, n) := \frac{\pi^d(\alpha,n) - \pi^c(\alpha,n)}{\pi^d(\alpha,n) - \pi^*(n)}$,*

*(ii) $\underline{\delta}(\alpha^-, n) = 0$, and*

may not be in other environments.

[8] In the examples below, and in many other setups, $\Pi^{\mathrm{M}}$ coincides with the standard monopoly profit.

*(iii)* $\frac{\partial \underline{\delta}(\alpha,n)}{\partial \alpha}$ *is monotonic in* $\alpha \in \mathcal{A}_n$ *if and only if* $1-\underline{\delta}(\alpha,n) > \frac{\frac{\partial \pi^c(\alpha,n)}{\partial \alpha}}{\frac{\partial \pi^d(\alpha,n)}{\partial \alpha}}$. *A necessary but not sufficient condition for this is given by* $\left|\frac{\partial \pi^d(\alpha,n)}{\partial \alpha}\right| > \left|\frac{\partial \pi^c(\alpha,n)}{\partial \alpha}\right|$.

*Proof* (i) It follows from (2), (ii) recall that $\pi^d(\alpha^-,n)=\pi^c(\alpha^-,n)=\pi^*(n)$ and apply the L'Hopital Rule: $\lim_{\alpha\to\alpha^-}\underline{\delta}(\alpha,n)=\underline{\delta}(\alpha^-,n)=0$. (iii) It follows from the fact that:

$$\text{sign}\left(\frac{\partial \underline{\delta}(\alpha,n)}{\partial \alpha}\right)=\text{sign}\left[\left(\frac{\partial \pi^d(\alpha,n)}{\partial \alpha}-\frac{\partial \pi^c(\alpha,n)}{\partial \alpha}\right)\left(\pi^d(\alpha,n)-\pi^*(n)\right)-\frac{\partial \pi^d(\alpha,n)}{\partial \alpha}\left(\pi^d(\alpha,n)-\pi^c(\alpha,n)\right)\right]. \quad (3)$$

First, $\underline{\delta}(\alpha,n)$ marks the range of discount factors for which the agreement $\alpha$ is implementable when there are $n$ firms in the market. Second, if the firms set $\alpha=\alpha^-$, the agreement is honored for any $\delta \geq 0$. In Cournot competition with homogeneous goods, for instance, choosing $\alpha^-$ means colluding on the very quantity the firms would produce competitively, so the agreement is self-enforcing by construction. The same logic applies in Bertrand price competition with differentiated products.

Third, consider a collusive agreement that moves away from $\alpha^-$ and toward $\alpha^+$. Both collusive and deviation profits rise, but the latter typically rises faster. The intuition is straightforward. When firms collude at $\alpha \approx \alpha^-$, industry profit is only at the competitive level; a marginal change in $\alpha$ has a first–order impact on both joint profit and the gain from deviation. By contrast, when firms collude near $\alpha \approx \alpha^+$, industry profit is almost maximized; a marginal change in $\alpha$ affects joint profit only to second order, yet it still produces a sizable first–order gain to a deviator. In short, the temptation to deviate intensifies as the agreement approaches $\alpha^+$. The condition in (iii), provides a ratio of change that guarantees that these temptations grow faster than collusive gains, in such a way that the minimum discount factor required to sustain the agreement rises as $\alpha$ moves from $\alpha^-$ toward $\alpha^+$.[9] These results suggest that collusive targets that yield higher joint profits are also harder to sustain —a tension that sits at the heart of our contribution and motivates the analysis in the next section.

## 2.1 Revising The Incentive Constraint

The comparative statics in the economic literature has mainly focused on the effect that factors, like the number of firms, have on the incentive compatibility constraint. That is, the analysis has been centered around $\frac{\partial \underline{\delta}(\alpha,n)}{\partial n}$ and, generally, around $\frac{\partial \underline{\delta}(\alpha^+,n)}{\partial n}$. However, for this procedure to be valid, certain constraints must be imposed on the model. For example, Bernheim & Whinston (1990)

[9]Note that the traditional Bertrand model is just on the boundary, and $\underline{\delta}$ is not monotonic in collusive price.

provides a multi-market contact analysis where firms can relax the no-cheating constraints (as long as there is some asymmetry among markets), facilitating the achievement of supracompetitive profits. This means, there is no change in supracompetitive profits but it can be obtained with lower discounts factors. Asker & Nocke (2021) point out that better observability of deviations "not only reduces the [minimum] discount factor but also increases the per-period value of each firm in the best equilibrium." In contrast to these papers, other factors, such as the number of firms, simultaneously influence both the set of implementable agreements and the associated gains in less clear or predictable ways.

We want to propose a procedure to assess the gains of collusion and its comparative static with respect to $n$; that is, the gains $\Delta$ that a firm gets with an implementable agreement $\alpha$ and how these gains change with $n$. The per period gain can be defined as the absolute gain $\Delta = \pi^c(\alpha, n) - \pi^*(n)$, or as the relative gain $\Delta^r = \frac{\pi^c(\alpha,n)}{\pi^*(n)} - 1$ of collusion. One possible way to achieve this is to consider the inverse of $\underline{\delta} = f(\alpha; n)$. Note that for any $\alpha \in \mathcal{A}_n$, there are many discount factors $\delta$ that satisfy the incentive constraint. Similarly, for any $\delta \in [0, 1]$, there are many $\alpha$ that satisfy the incentive constraint. Consequently, we propose to look for the inverse of a correspondence to obtain

$$\overline{\alpha} = f^{-1}(\underline{\delta}; n). \tag{4}$$

Our goal is to find the implementable outcome $\alpha$ that maximizes the gains from collusion. In fact, Equation (4) directly identifies the best collusive implementable outcome $\overline{\alpha}(\underline{\delta}, n) \in \mathcal{A}_n$ given that the discount factor of all firms is greater than $\underline{\delta}$. For clarity, we just write $\alpha(\delta, n)$ instead of $\overline{\alpha}(\underline{\delta}, n)$. Equation $\alpha(\delta, n)$ directly solves the cartel's problem, given the market characteristics: that the number of firms is $n$ and that all these firms discount factor is at least $\delta$. Note that the concepts differ between the minimum discount factor required, $\underline{\delta}$, and the actual (minimal) discount factor that all firms have, $\delta$. A higher $\underline{\delta}$ indicates that firms must be more patient to sustain collusion. In contrast, in Equation (4), a higher $\delta$ means that firms are indeed more patient. If the market has other relevant market characteristics, they will also be included in the Equation (4).

Replacing $\alpha(\delta, n)$ in the collusive gains functions, we find the best implementable gains of collusion in terms of the number of firms and the prevailing minimum discount factor, so as to get the expressions $\Delta(\delta, n)$ and $\Delta^r(\delta, n)$. We assume $|\alpha(\delta, n) - \alpha^-| > 0$ for all $\delta > 0$ and finite $n$; that is, an implementable and profitable collusive agreement exists whenever firms place any positive weight on future profits. This assumption rules out the traditional Bertrand case. Consequently, we can establish these straightforward results:

**Proposition 2** *In the market it is true that:*

*(i)* $\Delta^r(\delta, n) \geq 0$, $\forall \delta > 0$ *and finite* $n$,

*(ii)* $\Delta^r(\delta, n)$ *increases with* $\delta$,

*(iii)* $\Delta^r(\delta, n)$ *is increasing in* $n$ *if and only if* $-\frac{\pi_n^*(n)}{\pi^*(n)} > -\frac{\pi_n^c(\delta,n)}{\pi^c(\delta,n)}$.

First, there always exists an implementable agreement that increases the benefits of the colluding firms. Second, the relative gains of forming a cartel increase in $\delta$. As the incentive constraint is relaxed, there are more (or the same) implementable agreements, then the achievable profits increase. Third, and in simple terms, the proposition says that the relative gains of forming a cartel increase in the number of firms if and only if the relative profit loss under competition is greater than the relative profit loss under collusion. Given the definition of $\Delta^r$, the proof follows directly. This new, and maybe surprising, result is straightforward because the condition is relatively easy to satisfy.

We provide the economic intuition for the last result, both for low and high discount factors. Consider the case where the discount factor is high. In this setting, total industry profits from collusion may remain constant as the number of firms increases, but these profits must be shared among more participants. At the same time, profits under competition tend to fall due to stronger competitive pressure and are also divided among more firms. While the first effect is well recognized, the second effect—the decline in competitive profits and its distribution across more firms—is often overlooked in the literature. However, it is precisely this effect that drives our main result: typically the fall in competitive profits dominates, leading to higher relative gains from collusion as the number of firms increases.

When the discount factor is low, the analysis requires additional considerations. In this case, the no-cheating constraint becomes binding and limits the set of collusive outcomes that can be sustained: as the number of firms increases, this constraint becomes more restrictive—that is, $\alpha(\delta, n)$ becomes increasingly constrained as $n$ increases. This reduces the profits that firms can obtain through collusion. However, while this effect reduces the gains from collusion, it is generally of second-order importance for a given $\delta$, and it does not reverse the main result. The key driver remains the sharper drop in competitive profits, which continues to increase the relative gains from collusion as the number of firms rises. Actually, in all the cases analyzed so far, this relationship is monotonic in $n$ for any given $\delta > 0$.

We now extend the result for the case of absolute gains.

**Proposition 3** *In the market it is true that:*

*(i)* $\Delta(\delta, n)$ *is increasing in* $\delta$,

*(ii)* $\lim_{n\to\infty} \Delta(\delta, n) = 0$,

*(iii)* $\Delta(\delta, n)$ *increases in* $n$ *if and only if* $-\pi_n^*(n) > -\pi_n^c(\delta, n)$, *and*

*(iv)* $\Delta(\delta, n)$ *is maximized at* $n^*(\delta) \geq 1 \ \forall \delta > 0$.

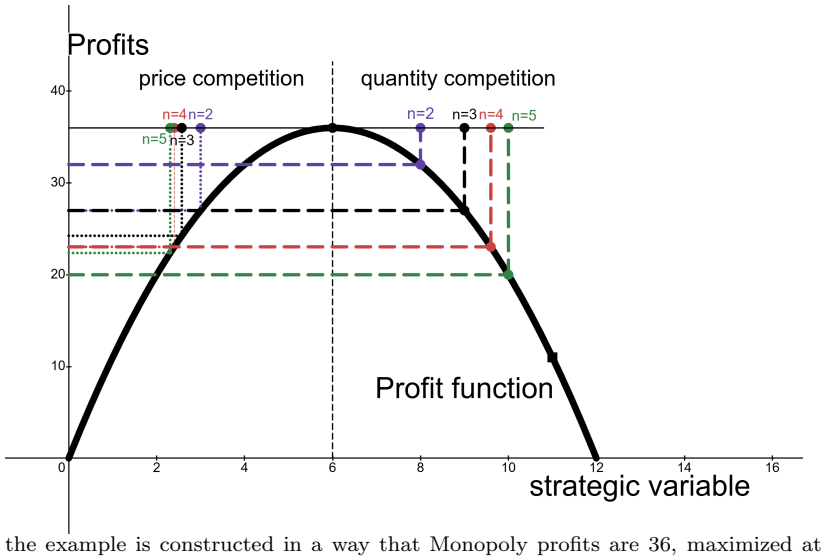
First, a higher discount factor $\delta$ relaxes the incentive constraint and raises $\pi^c(\delta, n)$ without affecting $\pi^*(n)$. Second, increasing the number of firms $n$ lowers both competitive and collusive profits; as $n \to \infty$, each tends to zero, so the absolute gain from collusion ultimately vanishes. This decline in absolute gains, however, is not monotonic. In fact, our third result shows that the absolute gain rises with $n$ whenever the profit lost from adding a firm under competition exceeds the corresponding loss under collusion.Finally, there is always a threshold $n^*$ beyond which the absolute gain from collusion falls as the number of firms increase.

The literature analyzes this problem as condition $-\pi_n^*(n) > -\pi_n^c(\delta, n)$ is never satisfied. However, whether or not it holds depends on the specific features of the market. In particular, the result varies with the type of competition and how much the entry of an additional firm increases competitive pressure. To build intuition, Figure 1 shows total industry profits as a function of a strategic variable. The example is constructed so that profits are maximized when the strategic variable is equal to 6, having an equivalent setup for either prices or quantities. Competition reduces prices and increases output compared to the level that maximizes total industry profits. We can interpret this same figure under either quantity or price competition with differentiated goods: under quantity competition, total output increases beyond 6; under price competition, prices fall below 6. The picture highlights with color the aggregated industry profits when there are 2, 3, 4, and 5 firms for both cases, using color purple, black, red, and green, respectively. Notice that with 4 firms (red lines), the aggregated profits under price and quantity competition are the same. However the implications of the models are different.

Under Cournot competition, total market output rises with the number of firms, which lowers total industry profits. For example, when there are 2 firms, the total output is 8; with 3 firms, it rises to 9, and so on. Notably, total profits decrease sharply when moving from 2 to 3 firms, satisfying the condition stated in the proposition. The same is true when moving from 3 to 4 firms. This pattern contributes to an increase in the gains from collusion as the number of firms increases. This effect dominates up to a certain threshold, above which the gains from collusion decrease in the number of firms. As noted in the introduction, comparing the potential gain of forming collusion in a two-firm market $(\Delta(2), \Delta^r(2)) = (\$2,\ 12.5\%)$, with those in an eleven-firm market $(\Delta(11), \Delta^r(11)) = (\$2.27,\ 227.27\%)$, offers a clear illustration of the results.

In the case of price competition, the result depends on the degree of product substitutability. In the example shown, competition between two firms is intense. However, adding more firms does not significantly increase competitive pressure. This is evident in the change in profits: when moving from 1 to 2 firms (from the maximum level to the purple line), profits drop sharply; but when moving from 2 to 3 firms (from the purple to the black line), profits experience a small reduction. The example assumes a high degree of substitution, so that the condition does not hold for $n \geq 2$. However, if products were less substitutable, the condition would be satisfied and the gains from collusion would again increase with the number of firms up to some threshold $n^*$.

**Figure 1:** Aggregated profits as a function of market strategic variable.




Note: the example is constructed in a way that Monopoly profits are 36, maximized at $p = Q = 6$. Under quantity competition, inverse demand is given by $p = \max\{0\,, 12 - Q\}$, with $Q = \sum_i q_i$. Under price competition, each firm demand is given by Equation 9 (in page 15) with $v = 12$ and $\gamma = 4$.

If the incentive no-cheating constraint is not binding, we are in the case highlighted by Werden & Baumann (1986). They focus in the Cournot model with linear inverse demand and constant marginal costs, and show that the absolute gains from collusion may increase with the number of firms. Our results provide a more general scope to these results, showing that they go beyond quantity competition and that they do not depend on the incentive constraint. Actually, we point out that the literature has relied in price competition with highly degree of product substitution and public information about constant marginal costs. We present below several traditional cases showing that $\Delta^r(\delta, n)$ increases in both $n$ and $\delta$. Moreover, $\Delta(\delta, n)$ has an inverted U-shaped function of $n$ and it is maximized at some $n^*(\delta)$. In fact, in many cases $n^*(\delta)$ is greater than 3 for all $\delta > 0$.

# 3 Example I: Cournot Competition

Consider $n$ symmetric firms competing à la Cournot, choosing quantities simultaneously. The market is characterized by a linear demand function $P = \max\{0, A - Q\}$, where $P$ is the price of the homogeneous good and $Q$ is the aggregate market output. We assume zero production costs for simplicity. In this setting, the strategic variable $\alpha$ introduced in Section 2 corresponds to the firm's individual quantity $q$, so that $\alpha = q$. More aggressive collusive targets are associated with lower values of $\alpha$. The collusive target $\alpha^+$ corresponds to the monopoly quantity, $\alpha^-$ corresponds to the competitive Cournot quantity, and $\underline{\delta}(q^c, n)$ denotes the minimum discount factor necessary to sustain a collusive agreement at quantity $q^c$. We summarize the equilibrium outcomes under competition, collusion at $q^c \in \mathcal{A}(n)$, and unilateral deviation in Table 1. Full derivations are available in Appendix A1.

**Table 1:** Cournot competition: Production, price and payoffs

| *Variable* | Competition | Collusion | Deviation |
|---|---|---|---|
| $q$ | $\frac{A}{n+1}$ | $q^c$ | $\frac{A-(n-1)q^c}{2}$ |
| $P$ | $\frac{A}{n+1}$ | $A - nq^c$ | $\frac{A-(n-1)q^c}{2}$ |
| $\pi$ | $\left(\frac{A}{n+1}\right)^2$ | $(A - nq^c)q^c$ | $\left(\frac{A-(n-1)q^c}{2}\right)^2$ |

Using the values in the table, we can compute $\mathcal{A}(n)$ and the critical discount factor $\underline{\delta}(q^c, n)$. For instance, $\mathcal{A}(n) = [A/(2n), A/(n+1)]$. Solving the incentive constraint, the best implementable collusive outcome for a given $\delta$ and $n$, denoted $q^c(\delta, n)$, is:

$$q^c(\delta, n) = \begin{cases} \dfrac{A}{n+1} \cdot \dfrac{\delta n^2 + 2\delta n - 3\delta - n^2 - 2n - 1}{\delta n^2 - 2\delta n + \delta - n^2 - 2n - 1} & 0 \leq \delta \leq \dfrac{n^2+2n+1}{n^2+6n+1}, \\ \dfrac{A}{2n} & \dfrac{n^2+2n+1}{n^2+6n+1} < \delta \leq 1. \end{cases} \tag{5}$$

We then define the relative gains from collusion as:

$$\Delta^r(\delta, n) = \begin{cases} \dfrac{4\delta(1-\delta)(n-1)^2(n+1)^2}{(\delta n^2 - 2\delta n + \delta - n^2 - 2n - 1)^2} & 0 \leq \delta \leq \dfrac{n^2+2n+1}{n^2+6n+1}, \\ \dfrac{(n-1)^2}{4n} & \dfrac{n^2+2n+1}{n^2+6n+1} < \delta \leq 1. \end{cases} \tag{6}$$

**Proposition 4** *In a symmetric Cournot model with linear demand and constant marginal cost:*

*(i)* $\Delta^r(0,n) = 0$*;*

*(ii)* $\Delta^r$ *does not depend on the market size parameter* $A$*;*

*(iii)* $\frac{\partial \Delta^r}{\partial \delta} \geq 0$*;*

*(iv)* $\frac{\partial \Delta^r}{\partial n} > 0$ *for any* $\delta > 0$*.*

The relative gain from collusion is a continuouus function and increase with both the discount factor and the number of firms. When $\delta = 0$, firms can only coordinate on the competitive quantity, so the relative gain is zero. As $\delta$ rises, firms can credibly commit to lower quantities and earn higher profits by colluding. Likewise, for any given $\delta > 0$, a larger number of firms makes the competitive benchmark less attractive, thereby amplifying the relative gains from collusion. Panel A of Figure 2 plots the relative–gain expression in equation (6) for several values of $n$.

In addition to the relative gains, we can also examine the absolute gains from collusion, denoted $\Delta(\delta, n)$. This measure captures the raw increase in profits from the best self-enforcing collusive outcome compared to the competitive benchmark. In the Cournot setting, the absolute gain takes the following form:

$$\Delta(\delta,n) = \begin{cases} \dfrac{4A^2\delta(1-\delta)(n-1)^2}{(\delta n^2 - 2\delta n + \delta - n^2 - 2n - 1)^2}, & \text{if} \quad 0 \leq \delta \leq \dfrac{n^2+2n+1}{n^2+6n+1}, \\ \dfrac{A^2(n-1)^2}{4n(n+1)^2}, & \text{if} \quad \dfrac{n^2+2n+1}{n^2+6n+1} < \delta \leq 1. \end{cases} \tag{7}$$

We summarize the behavior of $\Delta(\delta, n)$ in the following proposition.

**Proposition 5** *In a symmetric Cournot model with linear demand and constant marginal cost, the function* $\Delta(\delta, n)$ *satisfies:*

*(i)* $\Delta(0,n) = 0$*;*

*(ii)* $\frac{\partial \Delta}{\partial \delta} \geq 0$*;*

*(iii)* $\lim_{n\to\infty} \Delta(\delta, n) = 0$*;*

*(iv) For each* $\delta$*,* $\Delta(\delta, n)$ *is maximized at a critical number of firms* $n^*(\delta)$*, given by:*

$$n^*(\delta) = \begin{cases} \dfrac{1 + 2\sqrt{1-\delta} - \delta}{1-\delta}, & \text{if} \quad 0 \leq \delta \leq \dfrac{\sqrt{5}-1}{2}, \\ 2 + \sqrt{5}, & \text{if} \quad \dfrac{\sqrt{5}-1}{2} \leq \delta \leq 1. \end{cases} \tag{8}$$

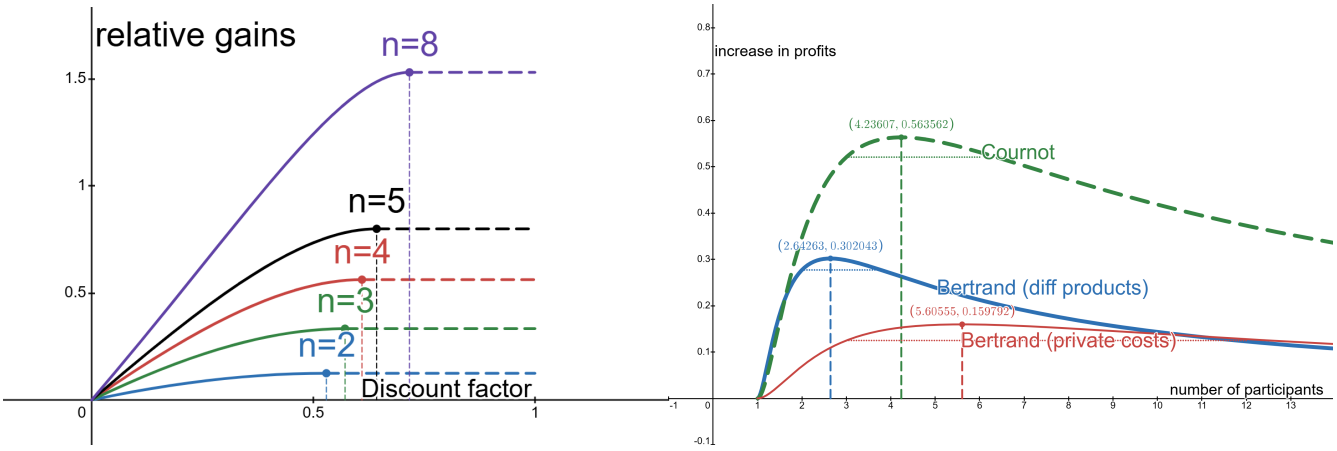
When firms are extremely impatient ($\delta = 0$), they can coordinate only on the competitive quantity, and the absolute gain from collusion is zero. As the discount factor rises, firms can credibly sustain lower outputs, so absolute gains increase. For any fixed $\delta > 0$, however, the relationship between the number of firms and the absolute gain is non-monotonic. Adding firms reduces per-firm collusive profits, yet it drives the competitive benchmark down even faster; profitability therefore first rises with $n$ and then falls. Hence each $\delta > 0$ admits a critical number of firms $n^*(\delta)$ at which $\Delta(\delta, n)$ is maximized. Panel B of Figure 2 illustrates this non-monotonic profile. The green dashed curve shows the Cournot case at a discount factor above $(\sqrt{5}-1)/2 \approx 0.62$; the peak occurs at $n^* \approx 4.23$, after which adding more firms reduces absolute gains.

Figure 3 plots the function $n^*(\delta)$ given in Equation 8. The relationship is weakly increasing in $\delta$, with $n^*(\delta)$ ranging from 3 to $2 + \sqrt{5} \approx 4.23$. Notably, even when firms are extremely impatient ($\delta = 0$), the maximum gains from collusion are achieved with more than two firms. This reinforces the idea that the profitability of collusion is not necessarily decreasing in the number of firms.

**Figure 2:** Cournot Competition: Gains from Collusion $\Delta^r(\delta, n)$ and $\Delta(\delta, n)$

Panel A. $\Delta^r(\delta, n)$ as a function of $\delta$

Panel B. $\Delta(\delta, n)$ as a function of $n$




*Source*: authors' calculations using $\Delta^r$ and $\Delta$ with $A = 1$ and $\gamma = 0.7$.

**Figure 3:** Critical number of firms for $\Delta(\delta, n)$


# 4 Example II: Bertrand Competition with Differentiated Products

Consider $n$ firms competing in prices à la Bertrand with differentiated goods and zero marginal cost. Demand follows the symmetric specification of Shubik & Levitan (2013):

$$q_i = \max\left\{0,\ \frac{1}{n}\left(v - p_i(1+\gamma) + \gamma\overline{P}\right)\right\} \tag{9}$$

Here, $q_i$ and $p_i$ denote the quantity demanded from and the price charged by firm $i$, respectively. The average market price is $\overline{P} = \frac{1}{n}\sum_{i=1}^{n} p_i$, and $\gamma \in (0, \infty)$ measures the degree of substitutability among products. As $\gamma$ increases, products become closer substitutes.[10] In this case, the strategic variable $\alpha$ corresponds to a firm's price, so that $\alpha = p$. More aggressive collusive targets are associated with higher values of $\alpha$. The upper collusive target $\alpha^+$ corresponds to the monopoly price—that is, the symmetric price that maximizes total industry profit and divides it equally among firms. Since the aggregate (market) demand has the same linear form as in our Cournot case (see Footnote 10), this upper collusive target is $p^c = \frac{v}{2}$. The lower collusive target $\alpha^-$ is defined as the price at which a firm is indifferent between colluding and deviating—that is, the lowest price that can be sustained without violating the incentive compatibility condition.

[10] Note that aggregate demand does not depend on $n$: $Q = \sum_{i=1}^{n} q_i = v - \overline{P}$. This is the same linear demand function used in the Cournot case, where $v$ plays the same role as the parameter $A$ in that model. The specification is conservative; if aggregate demand were to increase with $n$, the results would still hold and become even more striking.

As in the Cournot case, we summarize the equilibria under competition, collusion, and unilateral deviation in Table 2.

**Table 2:** Bertrand Competition: Prices, Output, and Profits

| *Variable* | Competition | Collusion | Deviation |
|---|---|---|---|
| $q$ | $\frac{v}{n} \cdot \frac{n+n\gamma-\gamma}{2n+n\gamma-\gamma}$ | $\frac{1}{n}(v-p^c)$ | $\frac{v+p^c\gamma\frac{n-1}{n}}{2n}$ |
| $p$ | $\frac{vn}{2n+n\gamma-\gamma}$ | $p^c$ | $\frac{v+p^c\gamma\frac{n-1}{n}}{2(1+\gamma\frac{n-1}{n})}$ |
| $\pi$ | $\frac{v^2(n+n\gamma-\gamma)}{(2n+n\gamma-\gamma)^2}$ | $\frac{p^c}{n}(v-p^c)$ | $\frac{(v+p^c\gamma\frac{n-1}{n})^2}{4n(1+\gamma\frac{n-1}{n})}$ |

Using the expressions in the table, we compute the critical discount factor ($\underline{\delta}(p^c, n, \gamma)$. Solving the incentive compatibility constraint, the best implementable collusive outcome for a given $\delta$, $n$, and $\gamma$, denoted $p^c(\delta, n, \gamma)$, is:

$$p^c(\delta, n, \gamma) = \begin{cases} \dfrac{vn}{2n+n\gamma-\gamma} \cdot \dfrac{H_1}{H_2}, & \text{if} \quad 0 \leq \delta \leq \delta_0(n,\gamma), \\ \dfrac{v}{2}, & \text{if} \quad \delta_0(n,\gamma) < \delta \leq 1, \end{cases} \tag{10}$$

where $\delta_0(n,\gamma) = \frac{(n\gamma-\gamma+2n)^2}{\gamma^2n^2-2\gamma^2n+\gamma^2+8\gamma n^2-8n\gamma+8n^2}$.

Expressions $H_1$ and $H_2$ (defined in Appendix A2.4) are polynomials with quadratic, linear, and constant terms in $n$ and $\gamma$, and linear and constant terms in $\delta$.[11] The threshold $\delta_0(n,\gamma)$ marks the level of patience beyond which the monopoly price—the upper collusive target $\alpha^+$—can be sustained as the best implementable outcome. For $\delta \leq \delta_0$, the implementable price lies on or below the monopoly level (i.e., $p^c(\delta, n, \gamma) \in [\alpha^-, \alpha^+]$).

We now turn to the relative gain from collusion, defined as the ratio of the incremental collusive profits over the competitive benchmark. Solving the incentive compatibility constraint and using the profit expressions from Table 2, we obtain:

$$\Delta^r(\delta, n, \gamma) = \begin{cases} \dfrac{4\delta\gamma^2(1-\delta)(n-1)^2(n\gamma-\gamma+2n)^2}{H_2^2}, & \text{if} \quad 0 \leq \delta \leq \delta_0(n,\gamma), \\ \dfrac{\gamma^2(n-1)^2}{4n(n+n\gamma-\gamma)}, & \text{if} \quad \delta_0(n,\gamma) < \delta \leq 1. \end{cases} \tag{11}$$

**Proposition 6** *In the Bertrand model with differentiated products, the relative gain from collusion $\Delta^r(\delta, n, \gamma)$ satisfies the following:*

[11] The reader is directed to the appendix for a detailed derivation of the results in this section.

*(i)* $\Delta^r(0, n, \gamma) = 0$*;*

*(ii)* $\Delta^r$ *is independent of the market size parameter* $v$*;*

*(iii)* $\frac{\partial \Delta^r}{\partial \delta} \geq 0$*;*

*(iv)* $\frac{\partial \Delta^r}{\partial n} \geq 0$*;*

*(v)* $\frac{\partial \Delta^r}{\partial \gamma} \geq 0$*.*

As in the Cournot case, the relative gain from collusion is a continuous function and increases with the discount factor: when firms are extrmemely impatient ($\delta = 0$), they can only coordinate on the competitive outcome, and no collusive profits are gained. But as $\delta$ rises, firms become more willing to forgo short-run deviations, allowing them to sustain higher collusive prices and extract greater profits. The gains also rise with the number of firms and the degree of product substitutability ($\gamma$), reflecting the fact that in more competitive environments—even with differentiation—the scope for profit improvement through collusion becomes larger. These patterns mirror the comparative statics derived earlier and further reinforce the idea that collusion may become more profitable as the number of firms grows.

We next consider the absolute gain from collusion relative to competition. These gains, denoted $\Delta(\delta, n, \gamma)$, capture the net profit improvement from sustaining collusion and depend on the number of firms, the discount factor, and the degree of product differentiation. The expression is:

$$\Delta(\delta, n, \gamma) = \begin{cases} \dfrac{4\delta\gamma^2 v^2 (1-\delta)(n-1)^2(n\gamma - \gamma + n)}{H_2^2}, & \text{if } 0 \leq \delta \leq \delta_0(n, \gamma), \\ \dfrac{\gamma^2 v^2 (n-1)^2}{4n(n\gamma - \gamma + 2n)^2}, & \text{if } \delta_0(n, \gamma) < \delta \leq 1. \end{cases} \tag{12}$$

**Proposition 7** *In a Bertrand model with differentiated products and linear demand, the absolute gain from collusion satisfies the following properties:*

*(i)* $\Delta(0, n) = 0$

*(ii)* $\frac{\partial \Delta}{\partial \delta} \geq 0$

*(iii)* $\lim_{n \to \infty} \Delta(\delta, n, \gamma) = 0$

*(iv)* $\Delta$ *has a critical* $n^*$ *at which the function reaches a maximum.*

The absolute gain is a continuous function and increases in $\delta$. When firms are extremely impatient ($\delta = 0$), collusion is limited to the competitive outcome, and the absolute gain is zero. As patience increases, firms can sustain higher collusive prices, so absolute gains rise. These gains are non-monotonic in the number of firms: for any fixed $\delta > 0$, there is a critical firm count $n^*(\delta, \gamma)$ at which collusion is most profitable. This non-monotonicity is a core implication of our framework—intermediate market structures can create the strongest incentives to collude, even with product differentiation. The blue dashed curve in Panel B of Figure 2 illustrates this behavior for $\gamma = 0.7$ and a high discount factor; the peak occurs at an interior value of $n$ ($n^* \approx 2.64$ in the figure).

The behavior of the optimal number of firms $n^*(\delta, \gamma)$ is shaped by both patience and substitutability. It rises with $\delta$, because more patient firms can sustain more aggressive collusive targets, and it falls with $\gamma$, because greater product substitutability increases the temptation to undercut rivals. Over a broad range of parameters, $n^*(\delta, \gamma)$ lies in the interval $[1, 3]$. Figure 3 plots $n^*(\delta, 1)$; in this benchmark $n^*$ changes little with $\delta$. More generally, $n^*$ is more sensitive to $\gamma$. As $\gamma \to \infty$, the model collapses to the standard Bertrand case with homogeneous goods, and $n^*(\delta, \gamma) \to 1$—reproducing the textbook result that collusion is harder to sustain as the number of firms grows.

Finally, while the absolute gain from collusion may diminish as product differentiation vanishes, the relative gain $\Delta^r$ continues to rise with the number of firms. That is, even when the setting approaches the classic Bertrand model with homogeneous goods, the percentage improvement over competitive profits can still grow. This divergence underscores our central finding: that collusive incentives—especially in relative terms—may strengthen as the number of firms increases, even in contexts traditionally viewed as least conducive to collusion.

# 5 Example III: Bertrand Competition with Uncertain Costs

We now examine price competition when marginal costs are private information, following Lofaro (2002).[12] Consider a market with $n$ symmetric firms producing a homogeneous good and facing linear demand

$$Q = \max\{0,\ A(1 - P)\}, \qquad A > 0.$$

All demand flows to the lowest-priced firm, with random tie-breaking.

[12]For a broader treatment of cost uncertainty in Bertrand models, see Spulber (1995).

At the beginning of each period, each firm $i$ draws a cost $c_i \sim U(0,1)$, observes it, and then posts a price while rivals' costs remain private. To collude, we assume firms may coordinate on a linear pricing rule

$$p_i \;=\; a^c + (1 - a^c)\, c_i,$$

where the parameter $a^c \in [0,1]$ serves as the collusive target. As shown in Appendix A3, the competitive benchmark is $a^c = 1/(n+1)$, while the most aggressive collusive target, $a^c = 1/2$, corresponds to monopoly pricing.

**Table 3:** Bertrand competition with uncertain costs: pricing rules and expected profits

| | *Competition* | *Collusion* | *Deviation* |
|---|---|---|---|
| $p_i$ | $\dfrac{1 + n\, c_i}{n+1}$ | $a^c + (1 - a^c)c_i$ | $\begin{cases} a^c, & c_i < \dfrac{a^c(n+1) - 1}{n}, \\ \dfrac{1 + n\, c_i}{n+1}, & c_i \geq \dfrac{a^c(n+1) - 1}{n} \end{cases}$ |
| $E[\pi_i]$ | $A\dfrac{n}{(n+1)^2(n+2)}$ | $A\dfrac{a^c(1 - a^c)}{n+2}$ | $A\dfrac{(1 - a^c)\left[(a^c)^2(n+1)^2 - 2a^c + 1\right]}{2n(n+2)}$ |

Using the expressions in Table 3, we first compute the critical discount factor $\underline{\delta}(a,n)$. Solving the resulting incentive-compatibility constraint gives the best implementable collusive outcome for a given $\delta$ and $n$, denoted $a(\delta,n)$:

$$a\,(\delta,n) = \begin{cases} \dfrac{n^2 + 3n + 2 - \delta\,(n^2 + n + 2) - \sqrt{\Psi}}{2(1-\delta)(n+1)^2}, & \text{if } 0 \leq \delta \leq \dfrac{n^2 + 2n + 1}{n^2 + 6n + 1}, \\ \dfrac{1}{2}, & \text{if } \dfrac{n^2 + 2n + 1}{n^2 + 6n + 1} < \delta \leq 1. \end{cases} \tag{13}$$

where

$$\Psi = \delta^2 n^4 + 6\delta^2 n^3 + \delta^2 n^2 - 8\delta^2 n - 2\delta n^4 - 8\delta n^3 + 2\delta n^2 + 8\delta n + n^4 + 2n^3 + n^2.$$

Although closed-form expressions for the absolute and relative gains, $\Delta(\delta,n)$ and $\Delta^r(\delta,n)$, can be derived, they are cumbersome and add little intuition; we therefore evaluate them numerically.

Our qualitative findings closely match the results obtained in the Cournot and differentiated-Bertrand cases. For the relative gain we have $\Delta^r(0,n) = 0$, $\partial\Delta^r/\partial\delta \geq 0$, and $\partial\Delta^r/\partial n \geq 0$. For the absolute gain, $\Delta(0,n) = 0$, $\partial\Delta/\partial\delta \geq 0$, and $\lim_{n\to\infty} \Delta(\delta,n) = 0$. When $\delta$ is below a threshold value

($\delta \approx 0.66$), the absolute gain peaks at a critical number of firms, $n^*(\delta) > 3.2$, and this peak shifts rightward as firms become more patient. Once $\delta$ exceeds this threshold, the maximizer stabilizes at $n^* = 2 + \sqrt{13} \approx 5.6$.[13] Panel B of Figure 2 illustrates the absolute gain curve for any $\delta$ above the threshold.

Hence, even with private information, the relative gain from collusion over competition, $\Delta^r$, increases monotonically with $n$: competition erodes expected profits faster than collusion, so the proportional reward from coordinating prices rises as the number of firms increases. By contrast, absolute gains are once again non-monotonic, peaking in markets with a moderately small number of firms. Thus, our central insight persists under cost uncertainty: a larger number of firms can strengthen the incentive to collude—particularly in relative terms—even when information is incomplete.

# 6 Conclusions

This paper re-examines how the number of firms affects cartel viability by isolating one often-overlooked channel: the gains available from collusive agreements that satisfy the incentive constraint. To keep that focus sharp, we abstract from forces that likely push in the opposite direction—coordination and monitoring costs, the design of punishment strategies, and exposure to antitrust risk. Although these frictions presumably rise with market size, our aim is to show that the *benefit* side of the ledger can, by itself, grow as more firms enter the market.

Our findings complement the standard critical-discount-factor approach. That approach remains valid under certain conditions, yet it does not preclude our broader conclusion that collusive incentives can increase with the number of firms. In particular, the benchmark Bertrand model with homogeneous products is a knife-edge case: its predictions fail under several plausible extensions—for example, when product differentiation is introduced, a feature that often emerges endogenously in real-world markets. When competition is softened by differentiation—whether horizontal, à la d'Aspremont *et al.* (1979), or vertical, à la Shaked & Sutton (1982)—our observation becomes even more relevant.

We emphasize both *absolute* and *relative* gains from collusion. Absolute gains eventually approach zero as the number of firms in the market increases, a purely mathematical consequence of profits vanishing in markets with arbitrarily large $n$. Relative gains, however, capture the rate

[13] This value maximizes $\Delta(\delta, n) = A\left[\frac{1}{4(n+2)} - \frac{n}{(n+1)^2(n+2)}\right]$, which corresponds to the absolute gain at the most aggressive collusive target, monopoly pricing ($a^c = 1/2$).

of return on the firm's "investment" in a cartel and remain informative even when absolute profits are small. They also fit settings where the cost of colluding is proportional to illicit profits. An extra $2 millions in collusive profit is far more compelling when competitive profit is $1 million than when it is $16 millions; relative gains make that distinction explicit.

Taken together, our analysis offers a simple theoretical rationale for empirical studies that document cartels with many participants. It also cautions policymakers against assuming that entry automatically disciplines collusion; the net effect depends on how quickly competitive profits erode relative to collusive profits once coordination is feasible.

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# Appendix

# A1 Analytical Details: Cournot Competition

## A1.1 Optimal Sales Decisions and The No-cheating Constraint

Given an inverse demand function $P = \max\{0, A - X\}$ we can obtain the competition, collusive, and deviation quantities and profits under Cournot competition.

- **Competition Equilibrium:** Firm $i$ solves the problem $\max_{q_i} \ \pi_i = \max_{q_i} \ P(X)q_i$; that is, $\max_{q_i}(A - X)q_i$. The first order condition is given by $A - X - q_i = 0$. By symmetry $q_i = q$, we get $q = \frac{A}{n+1}$. Then profits under competition are $\pi^*(n) = \left(\frac{A}{n+1}\right)^2$
- **Collusion:** A symmetric collusive agreement implies that all $n$ firms agree on producing a quantity $q^c$. The maximal value of $q^c$ is $\frac{A}{n+1}$ and the minimal is $\frac{A}{2n}$ (the one maximizing joint profits). Consequently, $q^c \in [\frac{A}{2n}, \frac{A}{n+1}]$, below we define the condition for $q^c$ to be implementable. Given an implementable $q^c$, each firm earns $\pi^c(n, q^c) = (A - nq^c)q^c$.
- **Deviation from Collusion:** Consider that the firm $i$ is planning to breach the collusive agreement while the $n-1$ remaining firms continue to honor it by sticking to the agreed level of sales $q^c$. The problem faced by the non-complier firm is given by: $\max_{q_i} \ \pi_i = \max_{q_i} \ (A - (n-1)q^c - q_i)\, q_i$. From the first order condition we get the best deviation $q_i = \frac{A-(n-1)q^c}{2}$. The deviating firm earns $\pi^d = \left(\frac{A-(n-1)q^c}{2}\right)^2$.

Using Equation for $\underline{\delta}$ we can calculate the minimum discount factor that allows for a collusive agreement where each firm produces $q^c$ to be stable:

$$\underline{\delta} = \frac{\left(\frac{A-(n-1)q^c}{2}\right)^2 - (A - nq^c)q^c}{\left(\frac{A-(n-1)q^c}{2}\right)^2 - \left(\frac{A}{n+1}\right)^2} \tag{14}$$

Which can be expressed as:

$$\underline{\delta}(q^c, n) = \frac{-An^2 - 2An - A + n^3q^c + 3n^2q^c + 3nq^c + q^c}{-An^2 - 2An + 3A + n^3q^c - n^2q^c - nq^c + q^c} \tag{15}$$

From this expression, we obtain equation (4), which defines the profit-maximizing implementable quantity, $q^c$. That is, $q^c(\delta, n)$ represents the minimum quantity that can be sustained as part of a collusive agreement when the discount factor must be at least $\delta$ and there are $n$ firms. It is given by:

$$q^c(\delta, n) = \begin{cases} \dfrac{A}{n+1} \cdot \dfrac{\delta n^2 + 2\delta n - 3\delta - n^2 - 2n - 1}{\delta n^2 - 2\delta n + \delta - n^2 - 2n - 1} & 0 \leq \delta \leq \dfrac{n^2 + 2n + 1}{n^2 + 6n + 1}, \\ \dfrac{A}{2n} & \dfrac{n^2 + 2n + 1}{n^2 + 6n + 1} < \delta \leq 1. \end{cases} \tag{16}$$

Given $n$, this function $q^c(\delta, n)$ is decreasing in $\delta$ (weakly). If $\delta \in [\frac{n^2+2n+1}{n^2+6n+1}, 1]$, $\frac{\partial q^c(\delta,n)}{\partial \delta} = 0$. If $\delta \in [0, \frac{n^2+2n+1}{n^2+6n+1}]$,

$$\frac{\partial q^c(\delta, n)}{\partial \delta} = -\frac{4A(n-1)(n+1)}{\left(\delta n^2 - 2\delta n + \delta - n^2 - 2n - 1\right)^2} < 0 \tag{17}$$

If we compute total quantity $Q^c = nq^c$, $Q^c$ is (weakly) increasing in $n$. Replacing $q^c(\delta, n)$ in $\pi^c\left(q^c(\delta), n\right) = \left(A - nq^c(\delta, n)\right) q^c(\delta, n)$ we obtain the maximum implementable profits. If $\delta \in [\frac{n^2+2n+1}{n^2+6n+1}, 1]$, $\pi^c\left(\delta, n\right) = \frac{A^2}{4n}$ and if If $\delta \in [0, \frac{n^2+2n+1}{n^2+6n+1}]$

$$\pi^c\left(\delta, n\right) = \frac{A^2}{\left(n+1\right)^2} \cdot \frac{(-\delta n^2 - 2\delta n + 3\delta + n^2 + 2n + 1)(3\delta n^2 - 2\delta n - \delta + n^2 + 2n + 1)}{\left(\delta n^2 - 2\delta n + \delta - n^2 - 2n - 1\right)^2} \tag{18}$$

Now we can calculate both the absolute gains $\Delta = \pi^c(\delta, n) - \pi^*(n)$ and relative gains $\Delta^r = \frac{\pi^c(\delta,n)}{\pi^*(n)} - 1$ of collusive agreements.

## A1.2 Collusion Gains

The relative gains from collusion compared to (imperfect) competition, denoted by $\Delta^r = \frac{\pi^c}{\pi^*} - 1$, measure the proportional increase in profits under collusion ($\pi^c$) relative to the non-collusion benchmark ($\pi^*$). It can be expressed by:

$$\Delta^r(\delta, n) = \begin{cases} \dfrac{4\delta(1-\delta)(n-1)^2(n+1)^2}{(\delta n^2 - 2\delta n + \delta - n^2 - 2n - 1)^2} & 0 \leq \delta \leq \dfrac{n^2+2n+1}{n^2+6n+1} \\ \dfrac{(n-1)^2}{4n} & \dfrac{n^2+2n+1}{n^2+6n+1} < \delta \leq 1 \end{cases} \tag{19}$$

We next prove the results in Proposition 4, that are the following: In a symmetric Cournot's model with linear demand and constant marginal cost it is true that: (i) $\Delta^r(0, n) = 0$. (ii) $\Delta^r$ is not a function of the market size parameter $A$. (iii) $\frac{\partial \Delta^r}{\partial \delta} \geq 0$. (iv) $\frac{\partial \Delta^r}{\partial n} \geq 0$, with strict inequality if $\delta > 0$.

Proposition 4 has different results. (i) says that competition is the only achievable outcome with extremely impatient agents. (ii) implies that the relative gains only depend on the number of firms but not on market size. (iii) Intuitively, the relative gains increase with the discount factor, as the set of implementable outcomes increases with $\delta$, and some of these outcomes may imply a greater profit for the firms. Finally, and most important, the relative gains increase with the number of firms, for any $\delta > 0$. Collusion is always a profitable activity and its rate of return is increasing in the number of firms.

*Proof* (i) and (ii) are direct. For (iii), notice that in the subdomain $\frac{n^2+2n+1}{n^2+6n+1} < \delta \leq 1$, this derivative is equal to zero. For $0 < \delta \leq \frac{n^2+2n+1}{n^2+6n+1}$:

$$\frac{\partial \Delta^r}{\partial \delta} = \frac{4(n-1)^2(n+1)^2[\delta(n^2+6n+1) - (n^2+2n+1)]}{[\delta(n^2-2n+1) - (n^2+2n+1)]^3} \tag{20}$$

This expression is positive if the following conditions hold: (a) $\delta \leq \frac{n^2+2n+1}{n^2+6n+1}$ and (b) $\delta \leq \frac{n^2+2n+1}{n^2-2n+1}$. Both conditions hold as $0 < \delta \leq \frac{n^2+2n+1}{n^2+6n+1}$.

For the last part (iv), notice that if $\delta$ satisfies $\frac{n^2+2n+1}{n^2+6n+1} < \delta \leq 1$, $\frac{\partial \Delta^r}{\partial n} = \frac{n^2-1}{4n^2} > 0$. For small values of $\delta$, i.e., $0 < \delta \leq \frac{n^2+2n+1}{n^2+6n+1}$, the derivative is

$$\begin{aligned}\frac{\partial \Delta^r}{\partial n} &= -\frac{16\delta(1-\delta)(n-1)(n+1)[\delta(n^2-2n+1)+(n^2+2n+1)]}{[\delta(n^2-2n+1)-(n^2+2n+1)]^3}, \\ &= -\frac{16\delta(1-\delta)(n-1)(n+1)[\delta(n-1)^2+(n+1)^2]}{[\delta(n-1)^2-(n+1)^2]^3}.\end{aligned}$$

which is positive, since $\delta \leq \frac{n^2+2n+1}{n^2-2n+1} = \frac{(n+1)^2}{(n-1)^2}$.

Now, we consider the absolute gains of collusion over competition ($\Delta = \pi^c - \pi^*$), which are given by:

$$\Delta(\delta, n) = \begin{cases} \dfrac{4A^2\delta(1-\delta)(n-1)^2}{(\delta n^2 - 2\delta n + \delta - n^2 - 2n - 1)^2} & 0 \leq \delta \leq \dfrac{n^2+2n+1}{n^2+6n+1} \\ \dfrac{A^2(n-1)^2}{4n(n+1)^2} & \dfrac{n^2+2n+1}{n^2+6n+1} < \delta \leq 1 \end{cases} \tag{21}$$

We prove here the results in Proposition 5: In a symmetric Cournot's model with linear demand and constant marginal cost it is true that: (i) $\Delta(0, n) = 0$, (ii) $\frac{\partial \Delta}{\partial \delta} \geq 0$, (iii) $\lim_{n\to\infty} \Delta(\delta, n) = 0$, and (iv) $\Delta(\delta, n)$ has a maximum at the number of firms given by:

$$n^*(\delta) = \begin{cases} \dfrac{1+2\sqrt{1-\delta}-\delta}{1-\delta} & 0 \leq \delta \leq \dfrac{\sqrt{5}-1}{2} \\ 2+\sqrt{5} & \dfrac{\sqrt{5}-1}{2} \leq \delta \leq 1 \end{cases} \tag{22}$$

Again, (i) says that competition is the only achievable course of action with extremely impatient agents. (ii) implies that these gains increase as the incentive constraint is relaxed. (iii) says that the absolute gains from collusion eventually must decrease with the number of firms, as both the profits with and without collusion converge to zero. Finally, (iv) says that the gains from collusion are non-monotonic in the number of firms, allowing an increase in firms benefit of collusion when the number of firms increases.

*Proof* (i) is direct. (ii) is implied by $\frac{\partial \Delta^r}{\partial \delta} \geq 0$. (iii) holds by L´Hopital rule. (iv) For low $\delta$ the first order condition generates equation $\delta = \frac{(n+1)(n-3)}{(n-1)^2}$, which is inverted in $n^*$. For high $\delta$ the derivative of 21 with respect to $n$ generates $n^* = 2 + \sqrt{5}$.

We can observe that $n^*$ is an increasing function of $\delta$, with $n^*(\delta) \in [3, 2+\sqrt{5}]$. To confirm that $n^*(\delta)$ does not converge to two, we highlight that $\lim_{\delta\to 0} n^* = 3$. Moreover, $\Delta(\delta, 5) > \Delta(\delta, 2)$, $\forall \delta > 0$.

Actually for low values of $\delta$ we can compute $\frac{\partial \Delta}{\partial \delta}$ and check how it changes with $n$ around $\delta \sim 0$. $\frac{\partial \Delta}{\partial \delta}(0, n) = \frac{4A^2(n-1)^2}{(n+1)^4}$ and check that $\frac{\partial \Delta}{\partial \delta}(0, 2) \leq \frac{\partial \Delta}{\partial \delta}(0, n)$ for $n = 3, 4, 5$.

# A2 Analytical Details: Bertrand Competition

## A2.1 Optimal Pricing Decisions and the No-cheating Constraint

We consider the model with no costs and where the demand functions with $n$ firms is defined by $q_i = \max\{0, \frac{1}{n}(v - p_i(1+\gamma) + \gamma\overline{P})\}$. Competitive, collusive, and deviations profits are computed as follows:

- **Competition Equilibrium:** Firm $i$ chooses its price to maximize its profits, i.e., it solves $\max_{p_i} p_i q_i = \max_{p_i} \frac{p_i}{n}(v - p_i(1+\gamma) + \gamma\overline{P})$. The first order condition with respect to $p_i$ is given by $\frac{1}{n}\left[v - p_i(1+\gamma) + \gamma\overline{P} + p_i\left(-1-\gamma+\frac{\gamma}{n}\right)\right] = 0$. Imposing symmetry, i.e., $p_i = p$ and, obviously, $p = \overline{P}$, the equilibrium is:
$$p = \frac{nv}{2n + \gamma n - \gamma}, \tag{23}$$
and each firm earns $\pi^*(n) = \frac{v^2(n+n\gamma-\gamma)}{(2n+n\gamma-\gamma)^2}$

- **Optimal Collusion:** A cartel agrees to coordinate prices on a level $p^c$. We can define that a reasonable set of prices is given by $p^c \in [\frac{nv}{2n+\gamma n-\gamma}, \frac{v}{2}]$. The lower bound is defined by the competitive price. The upper bound is defined by the maximization of joint profits; i.e., $\max\limits_{p_1,p_2,\dots,p_n} \sum_{i=1}^n \pi_i = \max\limits_{p_1,p_2,\dots,p_n} \sum_{i=1}^n p_i q_i$. By symmetry, i.e., $p_i = p\ \forall i$, we write $\sum_{i=1}^n \pi_i = \sum_{i=1}^n p\frac{v-p}{n} = p(v-p)$. Then, the symmetric price that maximizes joint profits is $p^c = \frac{v}{2}$ and, thus, each firm obtains $\pi^c = \frac{v^2}{4}$.

- **Deviation from Collusion:** Consider that the firm $i$ is planning to breach the collusion agreement while the $n-1$ remaining firms continue to honor it by sticking to the agreed price $p^c$. The problem faced by the non-complier firm is given by:
$$\max_{p_i}\ p_i\, q_i = \max_{p_i}\ \frac{p_i}{n}\left(v - p_i(1+\gamma) + \gamma\frac{p_i}{n} + \gamma\frac{n-1}{n}p^c\right) \tag{24}$$
The first order condition is given by $\frac{1}{n}[v - p_i(1+\gamma) + \gamma\frac{p_i}{n} + \gamma\frac{n-1}{n}p^c - p_i(1+\gamma-\frac{\gamma}{n})] = 0$. Implying that the best deviating price is:
$$p_i = \frac{v + \gamma\frac{n-1}{n}p^c}{2(1+\gamma\frac{n-1}{n})}, \tag{25}$$
generating deviating profits $\pi^d = \frac{(v+p^c\gamma\frac{n-1}{n})^2}{4n(1+\gamma\frac{n-1}{n})}$.

The minimum discount factor necessary for a collusive agreement $p^c$ to be feasible is given by:

$$\underline{\delta}(p^c, n) = \frac{\frac{(v+p^c\gamma\frac{n-1}{n})^2}{4n(1+\gamma\frac{n-1}{n})} - \frac{p^c}{n}(v-p^c)}{\frac{(v+p^c\gamma\frac{n-1}{n})^2}{4n(1+\gamma\frac{n-1}{n})} - \frac{v^2(n+n\gamma-\gamma)}{(2n+n\gamma-\gamma)^2}} \tag{26}$$

Which can be expressed as:

$$\underline{\delta} = \frac{(n\gamma-\gamma+2n)^2}{\gamma(n-1)} \cdot \frac{(n\gamma p^c - \gamma p^c + 2np^c - nv)}{\gamma^2n^2p^c - 2\gamma^2np^c + \gamma^2p^c + 2\gamma n^2p^c + 3\gamma n^2v - 2\gamma np^c - 3\gamma nv + 4n^2v} \tag{27}$$

Given $\underline{\delta}(p^c, n)$, we can find the expression $p^c(\delta, n)$, as in (4), which is given by

$$p^c(\delta,n) = \begin{cases} \dfrac{vn}{2n+n\gamma-\gamma} \cdot \dfrac{H_1}{H_2}, & \text{if} \quad 0 \leq \delta \leq \dfrac{(n\gamma-\gamma+2n)^2}{\gamma^2n^2-2\gamma^2n+\gamma^2+8\gamma n^2-8n\gamma+8n^2}, \\ \dfrac{v}{2}, & \text{if} \quad \dfrac{(n\gamma-\gamma+2n)^2}{\gamma^2n^2-2\gamma^2n+\gamma^2+8\gamma n^2-8n\gamma+8n^2} < \delta \leq 1. \end{cases} \tag{28}$$

Where expression $H_1$ and $H_2$ are defined in A2.4. For the rest of the abbreviations used, the reader is directed to that section.

Naturally, $p^c(\delta, n)$ is expected to increase in $\delta$ as a higher discount factor would allow to sustain collusive agreements with higher prices. While, $\frac{\partial p^c(\delta,n)}{\partial \delta}$ for $\delta \in [\frac{(n\gamma-\gamma+2n)^2}{\gamma^2n^2-2\gamma^2n+\gamma^2+8\gamma n^2-8n\gamma+8n^2}, 1]$, we obtain that for $\delta \in [0, \frac{(n\gamma-\gamma+2n)^2}{\gamma^2n^2-2\gamma^2n+\gamma^2+8\gamma n^2-8n\gamma+8n^2})$ the derivate is:

$$\frac{\partial p^c(\delta,n)}{\partial \delta} = \frac{4\gamma n v(n-1)[\gamma(n-1)+n][\gamma(n-1)+2n]}{{H_2}^2} \geq 0. \tag{29}$$

Next we can define

$$p^c(\delta,n) = \begin{cases} \dfrac{v^2(n+\gamma(n-1))H_1H_3}{(2n+\gamma(n-1))^2(H_2)^2}, & \text{if} \quad 0 \leq \delta \leq \dfrac{(n\gamma-\gamma+2n)^2}{\gamma^2n^2-2\gamma^2n+\gamma^2+8\gamma n^2-8n\gamma+8n^2}, \\ \dfrac{v^2}{4n}, & \text{if} \quad \dfrac{(n\gamma-\gamma+2n)^2}{\gamma^2n^2-2\gamma^2n+\gamma^2+8\gamma n^2-8n\gamma+8n^2} < \delta \leq 1. \end{cases} \tag{30}$$

Now we can calculate both the absolute gains $\Delta = \pi^c(\delta, n) - \pi^*(n)$ and relative gains $\Delta^r = \frac{\pi^c(\delta,n)}{\pi^*(n)} - 1$ of collusive agreements.

## A2.2 Collusion Gains

The relative gains of collusion over competition ($\Delta^r(\delta, n) = \frac{\pi^c(\delta,n)}{\pi^*(n)} - 1$) are given by:

$$\Delta^r(\delta,n) = \begin{cases} \dfrac{4\delta\gamma^2(1-\delta)(n-1)^2(n\gamma-\gamma+2n)^2}{{H_2}^2}, & \text{if} \quad 0 \leq \delta \leq \dfrac{(n\gamma-\gamma+2n)^2}{\gamma^2n^2-2\gamma^2n+\gamma^2+8\gamma n^2-8n\gamma+8n^2}, \\ \dfrac{\gamma^2(n-1)^2}{4n(n+n\gamma-\gamma)}, & \text{if} \quad \dfrac{(n\gamma-\gamma+2n)^2}{\gamma^2n^2-2\gamma^2n+\gamma^2+8\gamma n^2-8n\gamma+8n^2} < \delta \leq 1. \end{cases} \tag{31}$$

We define $\delta_0(n, \gamma) = \frac{(n\gamma-\gamma+2n)^2}{\gamma^2n^2-2\gamma^2n+\gamma^2+8\gamma n^2-8n\gamma+8n^2}$ for the threshold in the domian.

**Proposition 8** *(i)* $\Delta^r(0, n) = 0$, *(ii)* $\Delta^r$ *is not a function of the market size parameter* $v$. *(iii)* $\frac{\partial \Delta^r}{\partial \delta} \geq 0$ *(iv)* $\frac{\partial \Delta^r}{\partial n} \geq 0$, *(v)* $\frac{\partial \Delta^r}{\partial \gamma} \geq 0$.

Points (i)-(iv) provide the same results as in the Cournot model. Result (v) derives a new result, revealing that the degree of product substitution generates more incentives to collude in terms of the relative increase in profits. This parameter interacts positively with the number of firms, to increase the relative returns of collusion.

*Proof* (i)-(ii) are direct from the equation $\Delta^r(\delta, n, \gamma)$. For, (iii) notice that

$$\frac{\partial \Delta^r}{\partial \delta} = \begin{cases} -4\gamma^2(n-1)^2\left[\gamma(n-1)+2n\right]^2 \dfrac{H_4}{H_2^3}, & \text{if} \quad 0 \leq \delta_0(n,\gamma), \\ 0, & \text{if} \quad \delta_0(n,\gamma) < \delta \leq 1, \end{cases} \tag{32}$$

Notice that $H_4 \leq 0$ if $\delta \leq \frac{(n\gamma-\gamma+2n)^2}{\gamma^2(n-1)^2+8n[\gamma(n-1)+n]}$, which is exactly the domain limite. Also, $H_2 \geq 0$ if $\delta \leq \frac{(n\gamma-\gamma+2n)^2}{\gamma^2(n-1)^2}$, which holds since $\delta \leq \frac{(n\gamma-\gamma+2n)^2}{\gamma^2(n-1)^2+8n[\gamma(n-1)+n]} < \frac{(n\gamma-\gamma+2n)^2}{\gamma^2(n-1)^2}$.

For (iv) notice that

$$\frac{\partial \Delta^r}{\partial n} = \begin{cases} -16\delta\gamma^2(1-\delta)(n-1)\left[\gamma(n-1)+2n\right] \cdot \dfrac{H_5}{H_2^3}, & \text{if} \quad 0 \leq \delta \leq \delta_0(n,\gamma), \\ \dfrac{\gamma^2(n-1)\big(2n(n-1)+\gamma(2n(n-2)+1)\big)}{4n^2(n+n\gamma-\gamma)^2}, & \text{if} \quad \delta_0(n,\gamma) < \delta \leq 1. \end{cases} \tag{33}$$

We have shown earlier than $-H_2 \leq 0$. The proof that $H_5 \leq 0$ is identical. Then $\frac{\partial \Delta^r}{\partial n} > 0$ for $\delta > 0$, $n > 1$, $\gamma > 0$.

For (v) notice that

$$\frac{\partial \Delta^r}{\partial \gamma} = \begin{cases} -16\delta\gamma n(1-\delta)(n-1)^2\left[\gamma(n-1)+2n\right] \cdot \dfrac{H_5}{H_2^3}, & \text{if} \quad 0 \leq \delta \leq \delta_0(n,\gamma), \\ \dfrac{\gamma(n-1)^2\big(2n+\gamma(n-1)\big)}{4n(n+n\gamma-\gamma)^2}, & \text{if} \quad \delta_0(n,\gamma) < \delta \leq 1. \end{cases} \tag{34}$$

We have seen previously that both $-H_2$ and $H_5$ are negative expressions, then $\frac{\partial \Delta^r}{\partial \gamma} > 0$ for $\delta > 0$, $n > 1$, $\gamma > 0$.

## A2.3 Collusion: Absolute Gains

The absolute gains of collusion over competition ($\Delta(\delta, n) = \pi^c(\delta, n) - \pi^*(n)$) are given by:

$$\Delta(\delta, n) = \begin{cases} \dfrac{4\delta\gamma^2 v^2(1-\delta)(n-1)^2(n\gamma-\gamma+n)}{H_2^2} & 0 \leq \delta \leq \delta_0(,\gamma), \\ \dfrac{\gamma^2 v^2(n-1)^2}{4n(n\gamma-\gamma+2n)^2} & \dfrac{(n\gamma-\gamma+2n)^2}{\gamma^2n^2-2\gamma^2n+\gamma^2+8\gamma n^2-8n\gamma+8n^2} < \delta \leq 1 \end{cases} \tag{35}$$

**Proposition 9** *(i)* $\Delta(0, n) = 0$, *(ii)* $\frac{\partial \Delta}{\partial \delta} \geq 0$ *(iii)* $\lim_{n\to\infty} \Delta(\delta, n) = 0$ *(iv)* $\Delta$ *has a critical* $n^*$ *at which the function reaches a maximum. We define* $\delta_1(\gamma) = \frac{(\gamma+2)^2(1+\sqrt{4\gamma+9}\,)}{\sqrt{4\gamma+9}\;(\gamma^2+5\gamma+5)+5\gamma^2+19\gamma+17}$. *If* $\delta \geq \delta_1(\gamma)$, *then*

$$n^*(\delta, \gamma) = \frac{\gamma + \sqrt{4\gamma+9} + 3}{\gamma+2}. \tag{36}$$

*If* $\delta < \delta_1(\gamma)$, *then* $n^*$ *is implicitly defined by the cubic equation* $H_6 = 0$.

*Proof* (i) is direct, (ii) is implied by $\frac{\partial \Delta^r}{\partial \delta} \geq 0$. (iii) is obtained by the L'Hopital rule. (iv) If $\delta$ is high enough, such that the incentive constraint is not binding, the critical number of firms that maximizes $\Delta(\delta, n)$ is defined by

$$n^*(\delta, \gamma) = \frac{\gamma + \sqrt{4\gamma + 9} + 3}{\gamma + 2}. \tag{37}$$

Replacing this expression $n^*(\delta, \gamma)$ in the bound $\delta_0(, \gamma)$ we find the bound in terms of $\gamma$ defined as $\delta_1(\gamma) = \frac{(\gamma+2)^2(1+\sqrt{4\gamma+9})}{\sqrt{4\gamma+9}\ (\gamma^2+5\gamma+5)+5\gamma^2+19\gamma+17}$. Then, for $\delta_1(\gamma) \leq \delta \leq 1$[14], the critical number of firms is defined by (36).

However, if $0 \leq \delta \leq \delta_1(\gamma)$, there is no closed-form expression for $n^*$, but it is implicitly defined by the cubic equation $H_6 = 0$.

With numerical analysis (or a figure), it can be shown that $n^*$ is an increasing function of $\delta$, a decreasing function of $\gamma$ and that its range is given by:

$$1 \leq n^* \leq 3. \tag{38}$$

To confirm that $n^*(\delta, \gamma)$ does not converge to two $\forall \gamma$, we can highlight that the increase in profits as $\delta \to 0$ depends on $n$. $\lim_{\delta \to 0} \Delta(\delta, n, \gamma)$ is maximized at $n = 3$ for $\gamma < 1.13$.

## A2.4 Bertrand Competition Appendix Notation

$$H_1 = 3\delta\gamma^2n^2 - 6\delta\gamma^2n + 3\delta\gamma^2 + 4\delta\gamma n^2 - 4\delta n\gamma \quad + \gamma^2n^2 - 2\gamma^2n + \gamma^2 + 4\gamma n^2 - 4n\gamma + 4n^2 \tag{39}$$

$$H_2 = -\delta\gamma^2n^2 + 2\delta\gamma^2n - \delta\gamma^2 + \gamma^2n^2 - 2\gamma^2n + \gamma^2 + 4\gamma n^2 - 4n\gamma + 4n^2 \tag{40}$$

$$H_3 = -\delta\gamma^2n^2 + 2\delta\gamma^2n - \delta\gamma^2 - 4\delta\gamma n^2 + 4\delta n\gamma + \gamma^2n^2 - 2\gamma^2n + \gamma^2 + 4\gamma n^2 - 4n\gamma + 4n^2 \tag{41}$$

$$H_4 = \delta\gamma^2n^2 - 2\delta\gamma^2n + \delta\gamma^2 + 8\delta\gamma n^2 - 8\delta n\gamma + 8\delta n^2 - \gamma^2n^2 + 2\gamma^2n - \gamma^2 - 4\gamma n^2 + 4n\gamma - 4n^2 \tag{42}$$

$$H_5 = -\delta\gamma^2n^2 + 2\delta\gamma^2n - \delta\gamma^2 - \gamma^2n^2 + 2\gamma^2n - \gamma^2 - 4\gamma n^2 + 4n\gamma - 4n^2 \tag{43}$$

$$\begin{aligned} H_6 = {} & \delta\gamma^3n^3 - 3\delta\gamma^3n^2 + 3\delta\gamma^3n - \delta\gamma^3 + \delta\gamma^2n^3 - \delta\gamma^2n^2 - \delta\gamma^2n + \delta\gamma^2 - \gamma^3n^3 + 3\gamma^3n^2 - 3\gamma^3n + \gamma^3 \\ & - 5\gamma^2n^3 + 17\gamma^2n^2 - 19\gamma^2n + 7\gamma^2 - 8\gamma n^3 + 28\gamma n^2 - 20\gamma n - 4n^3 + 12n^2 \end{aligned} \tag{44}$$

[14] Observe that $0 \leq \frac{(\gamma+2)^2(1+\sqrt{4\gamma+9})}{\sqrt{4\gamma+9}\ (\gamma^2+5\gamma+5)+5\gamma^2+19\gamma+17} \leq 1$ for all values of $\gamma \geq 0$. We can also verify that: $\lim_{\gamma\to\infty} \frac{(\gamma+2)^2(1+\sqrt{4\gamma+9})}{\sqrt{4\gamma+9}\ (\gamma^2+5\gamma+5)+5\gamma^2+19\gamma+17} = 1$.

# A3 Analytical Details: Bertrand Competition with uncertain costs

## A3.1 Optimal Pricing Decisions

- **Competition Equilibrium**

  Firm $i$ competing in prices solves the problem

$$\max_{p_i} (p_i - c_i)(1 - p_i) Pr(p_i < \hat{p}_{-i}), \tag{45}$$

  where $\hat{p}_{-i}$ denotes the lowest price charged by the $n-1$ competitors of firm $i$. In a symmetric equilibrium $p_i(c_i) = p^*(c_i)$,; i.e., every firm chooses the same price given the same realization of unitary cost $c_i$. So the problem turns out to be:

$$\max_{p_i} (p_i - c_i)(1 - p_i)\left(1 - p^{*-1}(p_i)\right)^{n-1} \tag{46}$$

  Lofaro (2002) proves that the only symmetric Bayesian-Nash equilibrium is given by $p_i(c_i) = \frac{1+nc_i}{n+1}$. The realized benefit for cost $c_i$ is given by $\pi_i(c_i) = (p_i - c_i)(1 - p_i)^n \left(\frac{n+1}{n}\right)^{n-1}$, or:

$$\pi_i(c_i) = \frac{n(1 - c_i)^{n+1}}{(n+1)^2} \tag{47}$$

  Considering ex-ante, the expected benefits of competing is $E[\pi] = \frac{n}{(n+1)^2(n+2)}$.

- **Generalized Collusion**

  Given the form of the competitive pricing rule, a natural collusive agreement could consist of adopting the price rule:

$$p_i(c_i) = a^c + (1 - a^c)c_i \tag{48}$$

  for a given $a^c \in [\frac{1}{n+1}, \frac{1}{2}]$. Let define $\hat{c}_{-1}$ the lowest unit cost of the other $n-1$ firms. If $c_i < \hat{c}_{-1}$, we can replace the strategy $p_i(c_i)$ into the profit $(p_i - c_i)(1 - p_i)$. If $c_i < \hat{c}_{-i}$, the price is the lowest and the realized benefits are equal to:

$$\pi_i(c_i) = a^c(1 - a^c)(1 - c_i)^2 \quad \text{if} \quad c_i < \hat{c}_{-i} \tag{49}$$

  So, for each $c_i$ we can compute the probability that all the other costs are higher than this value. This is $c_{(1)}$, the first order statistic.

  The expected payoff is:

$$E(\pi_i) = E\left(E\left(\pi_i | c_i = \hat{c}_{-i}\right)\right) = \frac{1}{n} E\left(a^c(1 - a^c)\left(1 - c_{(1)}\right)^2\right) = \frac{a^c(1 - a^c)}{n}\frac{n}{n+2} = \frac{a^c(1 - a^c)}{n+2} \tag{50}$$

- **Optimal Collusion:** In an optimal collusion setting, firms follow the pricing rule of a monopolist[15]:

$$p_i = \frac{1 + c_i}{2} \tag{51}$$

[15] Another possible optimal collusive agreement could be the following: once the marginal costs are known, firms could identify the member with the lowest marginal cost and allow it to behave like a monopolist. Then, they could divide the profits evenly among the $n$ firms. Each firm would have a benefit equal to $\pi_i = \frac{1}{n}\left(\frac{1-c_{(1)}}{2}\right)^2$ It is clear that the expected benefit of this agreement is equal to the one we analyze in the main text.

As in the competitive setting, each firm will attract all the demand with probability $\frac{1}{n}$, but now with higher profits. Thus, benefits are equal to:

$$\pi_i = \left(\frac{1-c_i}{2}\right)^2 \quad if \quad c_i < \hat{c}_{-i} \tag{52}$$

and zero otherwise, where $\hat{c}_{-i}$ is the minimum marginal cost among the $n-1$ competitors. By the law of total expectation:

$$E(\pi_i) = E\left(E\left(\pi_i | c_i < \hat{c}_{-i}\right)\right) = \frac{1}{4n} E\left(\left(1-c_{(1)}\right)^2\right) = \frac{1}{4n}\frac{n}{n+2} = \frac{1}{4(n+2)} \tag{53}$$

Where $c_{(1)}$ denotes the first-order statistic of the marginal costs' distribution.

- **Deviation from Collusion** A firm $i$ could breach the collusive agreement and take advantage of the fact that the $n-1$ remaining firms continue to honor it, sticking to the agreed pricing rule coefficient $a^c$. Assume the non-complier firm decides to follow another pricing rule, breaching the collusion, before the marginal costs come to be known.

  Thus, the non-complier firm faces the problem:

$$\max_{p_i} \begin{cases} \left(p_i - c_i\right)\left(1-p_i\right) & p_i \leq a^c, \\ \left(p_i - c_i\right)\left(1-p_i\right)\left(1 - \dfrac{p_i - a^c}{1-a^c}\right)^{n-1} & p_i > a^c. \end{cases} \tag{54}$$

  This means that for each $c_i$ chooses the price that maximizes its expected profits.

  The FOC of the objective function in the sub-domain $p_i > a^c$ gives the same pricing rule of the competitive case $p_i = \frac{1+nc_i}{n+1}$. Notice that $p_i(c_i) > a^c$ if $c_i > \frac{a^c(n+1)-1}{n}$. For all $c_i \leq \frac{a^c(n+1)-1}{n}$, then $p_i(c_i) = a^m$. Since $a^c < \frac{1}{2} < \frac{1+c_i}{2}$, the optimal price is less than the monopoly price in the sub-domain $p_i < a^c$. Thus, the optimal strategy is to charge $p_i = a^c$ for all $c_i$ such that $p_i(c_i) \leq a^c$. Consequently, the optimal solution is given by:

$$p_i^d(c_i) = \begin{cases} a^c, & \text{if } c_i \leq \dfrac{a^m\left(n+1\right)-1}{n}, \text{ and} \\ \dfrac{1+nc_i}{n+1}, & \text{if } c_i > \dfrac{a^c\left(n+1\right)-1}{n}. \end{cases} \tag{55}$$

  Notice that the deviator sells with probability one if $c_i \leq \frac{a^m(n+1)-1}{n}$ and with probability $\left(\frac{n(1-c)}{(n+1)(1-a)}\right)^{n-1}$ if $c_i > \frac{a^m(n+1)-1}{n}$. This implies that expected benefits are equal to:

$$\pi_i(c_i) = \begin{cases} \left(1-a^c\right)\left(a^c - c_i\right), & \text{if } c_i \leq \dfrac{a^c\left(n+1\right)-1}{n}, \\ \dfrac{\left(1-a^c\right)\left(1-c_i\right)}{n+1}\left(\dfrac{n\left(1-c_i\right)}{\left(n+1\right)\left(1-a^c\right)}\right)^n, & \text{if } c_i > \dfrac{a^c\left(n+1\right)-1}{n}. \end{cases} \tag{56}$$

  The expected ex-ante payoff (before knowing $c_i$) is then [16]:

$$E(\pi_i) = \int_0^{\frac{a^c(n+1)-1}{n}} \left(1-a^c\right)\left(a^c - c_i\right) dc + \int_{\frac{a(n+1)-1}{n}}^1 \frac{\left(1-a^c\right)\left(1-c_i\right)}{n+1}\left(\frac{n\left(1-c\right)}{\left(n+1\right)\left(1-a^c\right)}\right)^n dc \tag{57}$$

$$E(\pi_i) = \frac{\left(1-a^c\right)\left[\left(a^c\right)^2\left(n+1\right)^2 - 2a^c + 1\right]}{2n(n+2)} \tag{58}$$

[16] Note that $0 \leq \frac{a(n+1)-1}{n} < 1$.

## A3.2 Collusion

The minimum discount factor necessary for a collusive agreement to be feasible $\underline{\delta}(a,n)$ is given by (we drop the superscript of $a^c$ for notation simplicity):

$$\underline{\delta} = \frac{E(\pi_i^d) - E(\pi_i^c)}{E(\pi_i^d) - E(\pi_i^*)}. = \frac{\frac{(1-a)\left[a^2(n+1)^2-2a+1\right]}{2n(n+2)} - \frac{a(1-a)}{n+2}}{\frac{(1-a)\left[a^2(n+1)^2-2a+1\right]}{2n(n+2)} - \frac{n}{(n+1)^2(n+2)}}. \tag{59}$$

$\underline{\delta}(a,n)$ can also be written as

$$\underline{\delta} = \frac{(1-a)(n+1)^2\left[a\left(n+1\right)-1\right]}{-a^2n^3-3a^2n^2-3a^2n-a^2+an^3+2an^2+3an+2a+n^2-2n-1}. \tag{60}$$

Observe that $\underline{\delta}\left(\frac{1}{n+1},n\right)=0$, so that an agreement that replicates competition can be trivially sustained. $\underline{\delta}(a,n)$ increase in $a$ for $a\in(\frac{1}{n+1},\frac{1}{2})$. In particular

$$\frac{\partial\underline{\delta}\left(a,n\right)}{\partial a} = \frac{2n\left(n+1\right)^2\left(a^2n^2+2a^2n+a^2-2an^2-2an+n^2+n-1\right)}{\left(-a^2n^3-3a^2n^2-3a^2n-a^2+an^3+2an^2+3an+2a+n^2-2n-1\right)^2} > 0. \tag{61}$$

From $\underline{\delta}(a,n)$ we can compute $a(\delta,n)$, which represents the maximal $a$ that can be sustained if the discount factor is greater than or equal to $\delta$ and the number of firms is $n$,

$$a\left(\delta,n\right) = \begin{cases} \dfrac{n^2+3n+2-\delta\left(n^2+n+2\right)-\sqrt{\Psi}}{2(1-\delta)(n+1)^2}, & \text{if } 0\leq\delta\leq\dfrac{n^2+2n+1}{n^2+6n+1}, \\ \dfrac{1}{2}, & \text{if } \dfrac{n^2+2n+1}{n^2+6n+1}<\delta\leq 1. \end{cases} \tag{62}$$

Where $\Psi=\delta^2n^4+6\delta^2n^3+\delta^2n^2-8\delta^2n-2\delta n^4-8\delta n^3+2\delta n^2+8\delta n+n^4+2n^3+n^2$.

Expressions $\Delta^r\left(\delta,n\right)$ and $\Delta\left(\delta,n\right)$ have closed-form expressions and can be found using:

$$E\left(\pi_i^c\right) = \frac{a\left(\delta,n\right)\left(1-a\left(\delta,n\right)\right)}{n+2}, \quad \& \quad E(\pi_i^*) = \frac{n}{\left(n+1\right)^2\left(n+2\right)}. \tag{63}$$

Due to the complexity of these cases, we examine them using numerical analysis. As the results closely align with those previously obtained, we present only a brief summary.

For the relative gains, we have that $\Delta^r(0,n)=0$, $\frac{\partial\Delta^r}{\partial\delta}\geq 0$ and $\frac{\partial\Delta^r}{\partial n}\geq 0$. For the absolute gains, we have that $\Delta(0,n)=0$, $\frac{\partial\Delta}{\partial\delta}\geq 0$, and $\lim_{n\to\infty}\Delta(\delta,n)=0$.

A critical number $n^*$ exists for which $\Delta\left(\delta,n\right)$ achieves a maximum. $n^*(\delta)$ is an increasing function of $\delta$ and $n^*(\delta)\in[3.21,2+\sqrt{13}]$. This means $\Delta(\delta,3)>\Delta(\delta,2)$ for all $\delta>0$. The value $n^*=2+\sqrt{13}$ holds if $\delta\geq\frac{22+6\sqrt{13}}{30+10\sqrt{13}}$ ( $\sim 0.6605$ ). $2+\sqrt{13}\in\arg\max_n\frac{1}{4(n+2)}-\frac{n}{(n+1)^2(n+2)}$ (i.e. $\arg\max_n\Delta\left(\delta,n\right)$ for $\delta\geq\frac{22+6\sqrt{13}}{30+10\sqrt{13}}$).